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Manifold structures for sets of solutions of the general relativistic constraint equations

Piotr T. Chruściel*, Erwann Delay[1]

Erwin Schrödinger Institute, Vienna, Austria

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Abstract

We construct manifold structures on various sets of solutions of the general relativistic initial data sets.

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1. Introduction

A natural question that arises in general relativity is whether some sets of solutions of the, say vacuum, constraint equations carry a manifold structure. For example, it is useful to have a Banach manifold structure on the set of asymptotically flat solutions of the constraint equations when trying to minimize the ADM mass [7,8,10]. Appropriate manifold structures allow one to use tools such as the Smale–Sard theorem, or the Baire category theorem, when discussing genericity of some properties of solutions of the Einstein equations. The existence of a *Fréchet* manifold structure[2] for (a subset of) the set of solutions of the vacuum constraint

* Corresponding author. Present address: Département de Mathématiques, Faculté des Sciences, Parc de Grandmont, F37200 Tours, France.
E-mail addresses: piotr@gargan.math.univ-tours.fr (P.T. Chruściel), delay@gargan.math.univ-tours.fr (E. Delay).
URL: http://www.phys.univ-tours.frpiotr.
[1] Present address: Département de Mathématiques, Faculté des Sciences, rue Louis Pasteur, F84000 Avignon, France.
[2] As pointed out to us by R. Bartnik, the argument of [20] leads to a loss of regularity of the metric, forcing one to work with C^∞ objects as far as the manifold structure is concerned.

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equations on a compact manifold is a consequence of linearization stability studies of Fischer, Marsden and Moncrief (see [20,21] and references therein, compare Theorem 4.6). The results there have also been studied in the context of asymptotically flat initial data sets on a manifold with compact interior in [1].

While the above results might be satisfactory for several purposes, they do *not* lead to a *Banach* manifold of solutions. In finite dimension there is no need to introduce a distinction between Banach, or Hilbert, or Fréchet manifold structure; however, the differences are significant in infinite dimension, because some facts which are true in Hilbert spaces are not necessarily true in all Banach spaces. Similarly some properties of Banach spaces do not carry over to Fréchet spaces. (The reader is referred to [18,24,25,29] for analysis on infinite dimensional manifolds.)

In the asymptotically flat case an alternative method has been developed by R. Bartnik[3] for constructing a Hilbert manifold structure on the space of solutions of the vacuum constraints, essentially based on the conformal method. It uses a (weighted) $H^k \times H^{k+1}$ topology, $k \geq 1$, on the space of (K, g)'s, and it is clear that the method generalizes to certain other settings of interest. Such spaces are well adapted to the evolution problem, at least for k large enough. However, in that method one does not have the flexibility in controlling boundary or asymptotic behavior which is provided by the Corvino–Schoen [12,15] version of the Fischer–Marsden–Moncrief approach.

The purpose of this note is to show that a *Banach* manifold structure can be obtained by a variation of the Fischer–Marsden–Moncrief-Corvino–Schoen method. It turns out that a direct application of the standard a priori estimates [28] for the construction below does not seem to lead to a manifold structure based on Sobolev spaces, which would have been more natural for the evolution problem, and which would have led to a *Hilbert* manifold structure. Instead, e.g. on compact manifolds without boundary, we obtain a manifold modeled on (a subset of) the space $C^{k,\alpha} \times C^{k,\alpha}$, $k \geq 4$, $\alpha \in (0, 1)$ of symmetric tensors. This appears somewhat surprising at first sight, as a natural set-up for the evolution problem (regardless of the Sobolev versus Hölder space issue) might seem to be one where the differentiability of the extrinsic curvature tensor K is one order less than that of the metric g. On the other hand, since K's can be thought of as variations of g's, from a manifold structure point of view it seems natural that the K's live in a space with the same differentiability as g. Whatever the natural space is, the $C^{k,\alpha} \times C^{k,\alpha}$ topology or weighted versions thereof are the ones which are obtained by the method here; this is a rather unexpected consequence of our analysis in this paper. As already pointed out, and as made clear in the applications below, the manifolds of initial data obtained here exhibit more structure than what is obtained by the conformal method and its variations.

In our construction of the manifold structure we use a smoothing device to recover the loss of regularity inherent to the Fischer–Marsden–Moncrief approach. This allows us to work consistently in spaces with finite differentiability, leading to the Banach manifold structure described above. We use a general approach of weighted spaces as in [12], which allows a simultaneous treatment of the compact case with or without boundary, and of

[3] R. Bartnik, in preparation; some similar ideas have also been considered by L. Andersson (private communication).

the asymptotically flat case, and of the conformally compactifiable case, with families of different topologies.

All the results presented below remain valid in the *time-symmetric* setting, $K = 0$. This implies that all the manifold structures presented here have their obvious counterparts for the set of Riemannian metrics with prescribed scalar curvature.

This paper is organized as follows: In Section 2 we review the framework of [12], and we show how the results there can be used to define a local Hilbert manifold of solutions near a given solution; the resulting manifolds can not be patched together in general because of insufficient regularity of the operators involved. In Section 3 we present our basic regularization procedure, which turns out to still be insufficient to provide a (global) Hilbert manifold structure. In Section 4 we therefore pass to an analysis in weighted Hölder spaces. We prove there that (the KID-free part of) the level sets of the constraints map are, globally, embedded submanifolds in a Banach space, see Theorem 5.2, under very general conditions on the weights; this is the main result of the paper. In fact, we prove that the level sets of the constraint map foliate (in a sense made precise in Remark 5.10) the KID-free part of the space of all (K, g)'s. In Section 6 we show that the hypotheses made in Theorem 5.2 are fulfilled on compact manifolds with or without boundary, or on asymptotically compactifiable manifolds, or on asymptotically flat manifolds. In Appendix A we prove a lemma which provides a submanifold structure in Banach spaces under rather general conditions, as well as a foliation result. In Appendix B we present two regularization procedures in weighted spaces, as needed in applications of the submanifold Theorem 5.2. For the convenience of the reader those results from [12] which play a key role in the current construction have been presented in detail, including some introductory comments borrowed from [12] whenever useful for the clarity of the argument.

2. The construction

Let

$$\mathscr{C}(K, g) := (J(K, g), \rho(K, g))$$

be the general relativistic *constraints map*:

$$\begin{pmatrix} J \\ \rho \end{pmatrix}(K, g) := \begin{pmatrix} 2(-\nabla^j K_{ij} + \nabla_i \operatorname{tr} K) \\ R(g) - |K|^2 + (\operatorname{tr} K)^2 - 2\Lambda \end{pmatrix}, \tag{2.1}$$

where Λ is the cosmological constant. (The function $c^4\rho/16\pi G$ is the energy density of the matter fields, while $c^4 J/16\pi G$ is the energy–momentum flux vector.) The general relativistic constraint equations are $\mathscr{C}(K, g) = 0$, whatever the space-dimension n. As those equations are trivial in space-dimension 0 and 1, in the remainder of this work we shall assume that $n \geq 2$.

Let $h = \delta g$ and $Q = \delta K$, the linearization $P_{(K,g)}$ of the constraints map at (K, g) reads

$$P_{(K,g)}(Q, h) = \begin{pmatrix} -K^{pq}\nabla_i h_{pq} + K^q{}_i(2\nabla^j h_{qj} - \nabla_q h^l{}_l) - 2\nabla^j Q_{ij} + 2\nabla_i \operatorname{tr} Q - 2(\nabla_i K^{pq} - \nabla^q K^p{}_i)h_{pq} \\ -\Delta(\operatorname{tr} h) + \operatorname{div}\operatorname{div} h - \langle h, \operatorname{Ric}(g)\rangle + 2K^{pl}K^q{}_l h_{pq} - 2\langle K, Q\rangle + 2\operatorname{tr} K(-\langle h, K\rangle + \operatorname{tr} Q) \end{pmatrix}. \tag{2.2}$$

Recall that a KID is defined as a solution (N, Y) of the set of equations $P^*_{(K,g)}(Y, N) = 0$, where $P^*_{(K,g)}$ is the formal adjoint of $P_{(K,g)}$:

$$P^*_{(K,g)}(Y,N) = \begin{pmatrix} 2(\nabla_{(i}Y_{j)} - \nabla^l Y_l g_{ij} - K_{ij}N + \operatorname{tr} K N g_{ij}) \\ \nabla^l Y_l K_{ij} - 2K^l{}_{(i}\nabla_{j)}Y_l + K^q{}_l \nabla_q Y^l g_{ij} \\ \quad - \Delta N g_{ij} + \nabla_i\nabla_j N + (\nabla^p K_{lp} g_{ij} - \nabla_l K_{ij})Y^l \\ \quad - N \operatorname{Ric}(g)_{ij} + 2NK^l{}_i K_{jl} - 2N(\operatorname{tr} K)K_{ij} \end{pmatrix}. \tag{2.3}$$

We shall denote by $\mathscr{K}(\Omega)$ the set of KIDs defined on an open set Ω. In vacuum space–times $(\mathscr{M}, g)$ KIDs on a spacelike hypersurface Ω are in one-to-one correspondence with Killing vectors of g [27] on the domain of dependence of Ω. A similar statement holds in electro-vacuum for appropriately invariant initial data for the electromagnetic field, the reader is referred to [9] for some comments about general matter fields.

Following [12], we will be using weighted function spaces defined as follows. Let ϕ and ψ be two smooth strictly positive functions[4] on M. For $k \in \mathbb{N}$ let $H^k_{\phi,\psi}(g)$ be the space of H^k_{loc} functions or tensor fields such that the norm[5]

$$\|u\|_{H^k_{\phi,\psi}(g)} := \left(\int_M \left(\sum_{i=0}^{k} \phi^{2i} |\nabla^{(i)} u|^2_g \right) \psi^2 \, \mathrm{d}\mu_g \right)^{1/2} \tag{2.4}$$

is finite, where $\nabla^{(i)}$ stands for the tensor $\underbrace{\nabla \cdots \nabla}_{i \text{ times}}$, with ∇—the Levi-Civita covariant derivative of g; we assume throughout that the metric is at least $W^{1,\infty}_{\mathrm{loc}}$; higher differentiability will be usually indicated whenever needed. For $k \in \mathbb{N}$ we denote by $\mathring{H}^k_{\phi,\psi}$ the closure in $H^k_{\phi,\psi}$ of the space of H^k functions or tensors which are compactly (up to a negligible set) supported in M, with the norm induced from $H^k_{\phi,\psi}$. The $\mathring{H}^k_{\phi,\psi}$'s are Hilbert spaces with the obvious scalar product associated to the norm (2.4). We will also use the following notation

$$\mathring{H}^k := \mathring{H}^k_{1,1}, \qquad L^2_\psi := \mathring{H}^0_{1,\psi} = H^0_{1,\psi},$$

so that $L^2 \equiv \mathring{H}^0 := \mathring{H}^0_{1,1}$. We set

$$W^{k,\infty}_\phi := \{u \in W^{k,\infty}_{\mathrm{loc}} \text{ such that } \phi^i |\nabla^{(i)} u|_g \in L^\infty\}$$

with the obvious norm, and with $\nabla^{(i)}u$—the distributional derivatives of u.

For ϕ and φ—smooth strictly positive functions on M, and for $k \in \mathbb{N}$ and $\alpha \in [0, 1]$, we define $C^{k,\alpha}_{\phi,\varphi}$ the space of $C^{k,\alpha}$ functions or tensor fields for which the norm

[4] We use the analysts' convention that a manifold M is always open; thus a manifold M with non-empty boundary ∂M does not contain its boundary; instead, $\bar{M} := M \cup \partial M$ is a manifold with boundary in the differential geometric sense. Unless explicitly specified otherwise *no* conditions on M are made—e.g. that ∂M, if non-empty, is compact—except that M is a smooth manifold; similarly no conditions, e.g. on completeness of (M, g), or on its radius of injectivity, are made.

[5] The reader is referred to [4,5,23] for a discussion of Sobolev spaces on Riemannian manifolds.

$$\|u\|_{C^{k,\alpha}_{\phi,\varphi}(g)} = \sup_{x\in M} \sum_{i=0}^{k} \left(\|\varphi\phi^i \nabla^{(i)} u(x)\|_g + \sup_{0\neq d_g(x,y)\leq \phi(x)/2} \varphi(x)\phi^{i+\alpha}(x) \frac{\|\nabla^{(i)}u(x) - \nabla^{(i)}u(y)\|_g}{d_g^\alpha(x,y)} \right)$$

is finite.

We will only consider weight functions with the property that there exists $\ell \in \mathbb{N} \cup \{\infty\}$ such that[6] for $0 \leq i < \ell$ we have

$$|\phi^{i-1}\nabla^{(i)}\phi|_g \leq C_i, \qquad |\phi^i\psi^{-1}\nabla^{(i)}\psi|_g \leq C_i, \qquad |\phi^i\varphi^{-1}\nabla^{(i)}\varphi|_g \leq C_i \tag{2.5}$$

for some constants C_i. The following situations seem to be of main interest:

- If M is compact without boundary we will use $\phi = \psi = \varphi = 1$.
- If ∂M is compact, smooth, and non-empty, we will use for $\phi = x$ a function which is a defining function for the boundary, at least in a neighborhood of the boundary; that is, any smooth non-negative function on $\bar{M}$ such that ∂M is precisely the zero-level set of x, with dx without zeros on ∂M. Then ψ and φ will be a power of x on a neighborhood of ∂M. Condition (2.5) will hold for metrics which are smooth up-to-boundary near ∂M.
- If M contains an asymptotically flat region, ϕ will behave as r, while φ and ψ will behave as a power of r in the asymptotically flat region; (2.5) will hold for a large class of asymptotically flat metrics.
- If M contains a conformally compactifiable region, then in a neighborhood of the conformal boundary ϕ will be taken to be 1, while ψ will be a power of the defining function of the conformal boundary.
- Exponentially weighted versions of the above will also be considered.

In all those situations one can obtain elliptic estimates in weighted spaces for the equations considered here by covering and scaling arguments together with the standard interior elliptic estimates on compact sets (cf., e.g. [2,3,6,11,22,26]). We will refer to this as *the scaling property*. More precisely, we shall say that *the scaling property* holds (with respect to some weighted Sobolev spaces with weight functions ψ and ϕ, and/or weighted Hölder spaces with weight functions φ and ϕ, whichever ones are being used will always be obvious from the context) if there exists a covering of M by a family of sets Ω_α, for α in some index set I, together with scaling transformations $\phi_\alpha : \Omega_\alpha \to \hat{\Omega}_\alpha$ on each of the sets Ω_α, such that the transformed fields $(\hat{K}_\alpha, \hat{g}_\alpha)$ on Ω_α are in[7] in $W^{3,\infty}(\hat{\Omega}_\alpha) \times W^{4,\infty}(\hat{\Omega}_\alpha)$, and such that the usual interior elliptic estimates on the $\hat{\Omega}_\alpha$'s can be pieced together to a weighted estimate, such as (4.7), for the original fields. Some sufficient conditions for the scaling property are discussed in [12, Appendix B]. We note that the scaling transformation of the fields on $\hat{\Omega}_\alpha$, $(K, g) \to (\hat{K}_\alpha, \hat{g}_\alpha)$, will typically consist of a pull-back of the fields, accompanied perhaps

[6] Conditions (2.5) will typically impose ℓ restrictions on the behavior of the metric and its derivatives in the asymptotic regions; it is therefore essential to allow $\ell < \infty$ if one does not wish to impose an infinite number of such conditions.

[7] It is conceivable that in some situations less a priori regularity on the $(\hat{K}_\alpha, \hat{g}_\alpha)$'s can be assumed, but this is the setup which seems to play the most important role in our paper; the reader should be able to adapt the differentiability conditions to his needs if required.

by a constant conformal rescaling. The "scaling property" is a condition both on the metric g, the extrinsic curvature tensor K, and on the weight functions involved: indeed, both the metric coefficients, the connection coefficients, as well as their derivatives, etc., which appear in our equations must have appropriate behavior under the above transformations so that the required estimates can be established.

Conditions (2.5) guarantee the following lemma.

Lemma 2.1. *Let $k \in \mathbb{Z}$, $k \geq -2$. Suppose that*[8] *$g \in W^{k+3,\infty}_{\mathrm{loc}}$ and that*

$$\mathrm{Ric}(g) \in \phi^{-2} W^{k+2,\infty}_{\phi}(g), \tag{2.6}$$

$$K \in \phi^{-1} W^{k+3,\infty}_{\phi}(g). \tag{2.7}$$

If (2.5) holds with $0 \leq i \leq k+2$, then the linear operators

$$P^*_{(K,g)} : \mathring{H}^{k+3}_{\phi,\psi}(g) \times \mathring{H}^{k+4}_{\phi,\psi}(g) \to \phi^{-1}\mathring{H}^{k+2}_{\phi,\psi}(g) \times \phi^{-2}\mathring{H}^{k+2}_{\phi,\psi}(g), \quad k \geq -2, \tag{2.8}$$

$$P_{(K,g)} : \psi^2(\phi\mathring{H}^{k+2}_{\phi,\psi}(g) \times \phi^2\mathring{H}^{k+2}_{\phi,\psi}(g)) \to \psi^2(\mathring{H}^{k+1}_{\phi,\psi}(g) \times \mathring{H}^{k}_{\phi,\psi}(g)), \quad k \geq 0, \tag{2.9}$$

are well defined, and bounded.

The following operator is of interest in our context,

$$L_{\phi,\psi} := \psi^{-2} P_{(K,g)} \Phi \psi^2 \Phi P^*_{(K,g)}, \tag{2.10}$$

where Φ is defined by

$$\Phi(x, y) := (\phi x, \phi^2 y). \tag{2.11}$$

A useful inequality to make things work is the following:

$$C\|\Phi P^*_{(K_0,g_0)}(Y, N)\|_{L^2_\psi(g_0)} \geq \|Y\|_{L^2_\psi(g_0)} + \|N\|_{\mathring{H}^1_{\phi,\psi}(g_0)}. \tag{2.12}$$

Let $\mathcal{K}_0$ be kernel of

$$\Phi P^*_{(K_0,g_0)} : \mathring{H}^1_{\phi,\psi}(g_0) \times \mathring{H}^2_{\phi,\psi}(g_0) \to L^2_\psi(g_0) \times L^2_\psi(g_0),$$

and let $\mathcal{K}_0^{\perp_{g_0}}$ be its $L^2_\psi(g_0) \oplus L^2_\psi(g_0)$-orthogonal. We denote by $\pi_{\mathcal{K}_0^{\perp_g}}$ the $L^2_\psi(g)$ projection onto $\mathcal{K}_0^{\perp_g}$. The following result, proved[9] in [12], is a weighted equivalent of those in [13,14,16] (compare [15]):

[8] The local differentiability conditions follow from the requirement that the $k+$ fourth covariant derivatives of N and the $k+$ third ones of Y can be defined in a distributional sense; both of those conditions are fulfilled by a metric $g \in W^{k+3,\infty}_{\mathrm{loc}}$—the reader should note that the first covariant derivatives of N do not involve the Christoffel symbols of g since N is a function.

[9] Actually in [12] the hypothesis is made that the cosmological constant Λ vanishes, which is not assumed here (compare (2.1)). The inclusion of a cosmological constant does not require any modifications of the proofs there, insofar as the results discussed here are concerned.

Theorem 2.2 ([12, Theorem 3.6]). *Let $k \geq 0$, $g_0 \in W^{k+4,\infty}_{\mathrm{loc}}$, suppose that (2.5) holds with $0 \leq i \leq 4+k$, that*

$$\mathrm{Ric}(g_0) \in \phi^{-2} W^{k+2,\infty}_{\phi}(g_0), \qquad K_0 \in \phi^{-1} W^{k+3,\infty}_{\phi}(g_0),$$

and that the weights ϕ and ψ have the scaling property. *If there exists a compact set $\mathscr{K} \subset M$ such that for all $\mathring{H}^1_{\phi,\psi}(g_0)$ vector fields Y and $\mathring{H}^2_{\phi,\psi}(g_0)$ functions N, both supported in $M\setminus\mathscr{K}$, the inequality (2.12) holds, then for all (K, g) close to (K_0, g_0) in $\phi^{-1}W^{k+3,\infty}_{\phi}(g_0) \times W^{k+4,\infty}_{\phi}(g_0)$ norm, the map*

$$\pi_{\mathcal{K}_0^{\perp_g}} L_{\phi,\psi} : \mathcal{K}_0^{\perp_g} \cap (\mathring{H}^{k+3}_{\phi,\psi}(g) \times \mathring{H}^{k+4}_{\phi,\psi}(g)) \to \mathcal{K}_0^{\perp_g} \cap (\mathring{H}^{k+1}_{\phi,\psi}(g) \times \mathring{H}^{k}_{\phi,\psi}(g))$$

is an isomorphism such that the norm of its inverse is bounded independently of (K, g).

We will see how to use this result, and certain variations thereof, to obtain a manifold structure on various sets of solutions of the vacuum constraint equations. More generally, one obtains a manifold structure on the set of initial data with (J, ρ)-fixed. Recall that a *Banach manifold* is a Hausdorff topological space M such that for every $p \in M$ there exists a neighborhood $\mathscr{U}_p \subset M$ of p equipped with a homeomorphism ϕ_p from $\mathscr{U}_p$ to an open subset $\mathscr{O}_p$ of some Banach space B_p. The couple $(\mathscr{U}_p, \phi_p)$, and sometimes simply the set $\mathscr{U}_p$, will be called a *coordinate patch*. On overlapping coordinate patches $\mathscr{U}_p$ and $\mathscr{U}_q$ the maps $\phi_p \circ \phi_q^{-1}$ are supposed to be smooth diffeomorphisms from their domains $\phi_q(\mathscr{U}_p \cap \mathscr{U}_q) \subset B_q$ to their images $\phi_p(\mathscr{U}_p \cap \mathscr{U}_q) \subset B_p$. One can similarly define the notion of a *Hilbert* manifold, and of a *Fréchet* manifold.

A connected embedded submanifold of an open subset of a Banach space is always a manifold modeled on any of its tangent spaces (which are all necessarily diffeomorphic, compare Corollary 5.11). We will actually prove that the level sets of the constraint map form embedded submanifolds in such a space, which will provide the desired manifold structure.

We start with the following observation.

Proposition 2.3 (Local Hilbert manifold (and submanifold)). *Under the hypotheses of Theorem 2.2, suppose that there are no KIDs*:

$$\mathring{H}^{k+3}_{\phi,\psi}(g_0) \times \mathring{H}^{k+4}_{\phi,\psi}(g_0) \supset \mathcal{K}_0 = \{0\}. \tag{2.13}$$

Assume that the map

$$\begin{aligned} &\psi^2(\phi\mathring{H}^{k+2}_{\phi,\psi}(g_0) \times \phi^2\mathring{H}^{k+2}_{\phi,\psi}(g_0)) \to \mathring{H}^{k+1}_{\phi,\psi}((g_0)) \times \mathring{H}^{k}_{\phi,\psi}((g_0)), \\ &(\delta K, \delta g) \mapsto \psi^{-2}\left\{ \begin{pmatrix} J \\ \rho \end{pmatrix} [(K_0, g_0) + (\delta K, \delta g)] - \begin{pmatrix} J \\ \rho \end{pmatrix} (K_0, g_0) \right\} \end{aligned} \tag{2.14}$$

is differentiable at zero. Then the set

$$\begin{aligned} \mathcal{S}_0 = \Big\{ &(Q, h) \in \psi^2(\phi\mathring{H}^{k+2}_{\phi,\psi}(g_0)\phi^2\mathring{H}^{k+2}_{\phi,\psi}(g_0)), \\ &(J, \rho)(K_0 + Q, g_0 + h) - (J, \rho)(K_0, g_0) = 0 \Big\} \end{aligned} \tag{2.15}$$

is an embedded submanifold of $\psi^2(\phi\mathring{H}^{k+2}_{\phi,\psi}(g_0) \times \phi^2\mathring{H}^{k+2}_{\phi,\psi}(g_0))$ in a neighborhood of zero.

Proof. A rather general justification is provided by Lemma A.1, Appendix A, by setting $u = Du(0) = \psi^2\Phi^2 P^*_{(K_0,g_0)}$, $v = \psi^{-2}[(J,\rho)(K_0+\cdot, g_0+\cdot) - (J,\rho)(K_0,g_0)]$, $Dv(0) = \psi^{-2}P_{(K_0,g_0)}$, $E = \mathring{H}^{k+3}_{\phi,\psi}(g_0) \times \mathring{H}^{k+4}_{\phi,\psi}(g_0)$, $F = \psi^2\Phi(\mathring{H}^{k+2}_{\phi,\psi}(g_0) \times \mathring{H}^{k+2}_{\phi,\psi}(g_0))$, and $G = \mathring{H}^{k+1}_{\phi,\psi}(g_0) \times \mathring{H}^{k}_{\phi,\psi}(g_0)$. The linear map u is continuous by Lemma 2.1, hence differentiable, while v is differentiable by hypothesis. The required isomorphism property for $Dv(0)\circ Du(0)$ follows from Theorem 2.2. □

Example 2.4. To motivate and illustrate the results so far, consider the case of a compact manifold M without boundary; other examples of interest will be treated in detail in Section 6. We choose $\phi = \psi \equiv 1$, so the spaces involved are standard Sobolev spaces. As M is compact we can take $\mathscr{K} = M$ so that condition (2.12) is trivially satisfied. The smoothness hypotheses on the map (3.2) are satisfied by standard calculus in Sobolev spaces. If (K_0, g_0) is a $C^{k+4} \times C^{k+3}$ initial data set without KIDs, Proposition 2.3 provides a manifold of $H^{k+2} \times H^{k+2}$ solutions of the constraint equations passing through (K_0, g_0).

Example 2.4 clearly exhibits an unfortunate differentiability mismatch, which leads to an essential obstruction when trying to glue together the coordinate patches obtained so far, and which therefore prevents one from obtaining a Hilbert manifold structure on the set of all solutions of the constraints equations using the method above. In Section 5 we will show that a (different) manifold structure can nevertheless be obtained using the following approach: consider a data set (K_1, g_1) without KIDs in an appropriate Hölder differentiability class, let $(\delta K_1, \delta g_1)$ be a solution of the constraints with the same differentiability class, and suppose that you can smooth out $(K_1 + \delta K_1, g_1 + \delta g_1)$ to a *smooth* set (K, g), in a way consistent with the set-up of Theorem 3.1. If one can solve the equation

$$\begin{pmatrix} J \\ \rho \end{pmatrix}((K_0+\delta K, g_0+\delta g) + \psi^2\Phi^2 P^*_{(K,g)}(Y,N)) - \begin{pmatrix} J \\ \rho \end{pmatrix}(K_0+\delta K, g_0+\delta g) = \begin{pmatrix} \delta J \\ \delta\rho \end{pmatrix}, \tag{2.16}$$

then one has a better chance of ending in a space with the original differentiability.

3. A regularized problem

The aim of this section is to implement the above in weighted Sobolev spaces. Consider again a compact manifold without boundary, let (K, g) be of $C^{k+2,\alpha} \times C^{k+2,\alpha}$ differentiability class, and first regularize (K, g) by the usual convolution method to obtain a new smooth couple $(K_\varepsilon, g_\varepsilon)$, then define

$$P^*_\varepsilon := P^*_{(K_\varepsilon, g_\varepsilon)}.$$

Roughly speaking, the equation we will attempt to solve will be (2.16) with $P^*_{(K,g)}$ there replaced by P^*_ε. The idea is to solve that equation for $(Y, N) \in C^{3,\alpha} \times C^{4,\alpha}$, and then

use elliptic regularity to get to $(Y, N) \in C^{k+3,\alpha} \times C^{k+4,\alpha}$, obtaining the desired extrinsic curvature and metric in $C^{k+2,\alpha} \times C^{k+2,\alpha}$; we emphasize that one would *not* get that last differentiability without the regularization. Somewhat more generally, let $0 \le k_0 \le k-2$, still in the compact boundaryless case one has the following convergence property, for $k \ge 2$:

$$P_\varepsilon^* \to P^* \text{ in } L(\mathring{H}^{k_0+3}_{\phi,\psi}(g_0) \times \mathring{H}^{k_0+4}_{\phi,\psi}(g_0), \phi^{-1}\mathring{H}^{k_0+2}_{\phi,\psi}(g_0) \times \phi^{-2}\mathring{H}^{k_0+2}_{\phi,\psi}(g_0)),$$

which is precisely what is needed to make the arguments work. In general we will therefore assume that we have a smoothing operation $(K, g) \to (K_\varepsilon, g_\varepsilon)$ such that

$$(K_\varepsilon, g_\varepsilon) \to_{\varepsilon\to 0} (K, g) \text{ in } \phi^{-1}W^{k_0+3,\infty}_\phi(g_0) \times W^{k_0+4,\infty}_\phi(g_0). \tag{3.1}$$

(In Appendix B we give conditions on the weight functions which guarantee that the smoothing operation with the above properties exists.) This leads to the following variation of Theorem 3.9 of [12].

Theorem 3.1. *Let $k \ge 0$, $g_0 \in W^{k+4,\infty}_{\mathrm{loc}}$, assume that (3.1) holds with $k_0 = k$ and suppose that (2.5) holds with $0 \le i \le 4+k$. Assume moreover that*

$$\mathrm{Ric}(g_0) \in \phi^{-2}W^{k+2,\infty}_\phi(g_0), \qquad K_0 \in \phi^{-1}W^{k+3,\infty}_\phi(g_0),$$

and that the weights ϕ and ψ have the scaling property. *Suppose further that there exists a compact set $\mathscr{K} \subset M$ such that for all $\mathring{H}^1_{\phi,\psi}(g_0)$ vector fields Y and $\mathring{H}^2_{\phi,\psi}(g_0)$ functions N, both supported in $M \setminus \mathscr{K}$, the inequality (2.12) holds. If the weights are such that the map*

$$\begin{aligned}
&\mathcal{K}_0^{\perp_g} \cap (\mathring{H}^{k+3}_{\phi,\psi}(g) \times \mathring{H}^{k+4}_{\phi,\psi}(g)) \to \mathcal{K}_0^{\perp_g} \cap (\mathring{H}^{k+1}_{\phi,\psi}(g) \times \mathring{H}^{k}_{\phi,\psi}(g)),\\
&(Y, N) \mapsto \pi_{\mathcal{K}_0^{\perp_g}} \psi^{-2} \left\{ \begin{pmatrix} J \\ \rho \end{pmatrix} [(K, g) + \psi^2\Phi^2 P^*_\varepsilon(Y, N)] - \begin{pmatrix} J \\ \rho \end{pmatrix} (K, g) \right\}
\end{aligned} \tag{3.2}$$

is differentiable in a neighborhood of zero, then it is bijective in a (perhaps smaller) neighborhood of zero, for all ε small enough. In other words, for sufficiently small ε, there exists $\delta > 0$ such that for all (K, g) close to (K_0, g_0) in $\phi^{-1}W^{k+3,\infty}_\phi(g_0) \times W^{k+4,\infty}_\phi(g_0)$, and for all pairs $(\delta J, \delta\rho) \in \psi^2(\mathring{H}^{k+1}_{\phi,\psi}(g) \times \mathring{H}^{k}_{\phi,\psi}(g))$ with norm less than δ, there exists a solution

$$(\delta K, \delta g) = \Phi\psi^2\Phi P^*_\varepsilon(Y, N) \in \psi^2(\phi H^{k+2}_{\phi,\psi}(g) \times \phi^2 H^{k+2}_{\phi,\psi}(g)), \tag{3.3}$$

close to zero, of the equation

$$\pi_{\mathcal{K}_0^{\perp_g}} \psi^{-2} \left\{ \begin{pmatrix} J \\ \rho \end{pmatrix} (K + \delta K, g + \delta g) - \begin{pmatrix} J \\ \rho \end{pmatrix} (K, g) \right\} = \pi_{\mathcal{K}_0^{\perp_g}} \psi^{-2} \begin{pmatrix} \delta J \\ \delta\rho \end{pmatrix}. \tag{3.4}$$

The solutions of the form (3.3) with sufficiently small norm are unique.

Proof. Instead of (3.4) *we consider the projection of the equation*

$$\begin{pmatrix} J \\ \rho \end{pmatrix} ((K, g) + \psi^2\Phi^2 P^*_\varepsilon(Y, N)) - \begin{pmatrix} J \\ \rho \end{pmatrix} (K, g) = \begin{pmatrix} \delta J \\ \delta\rho \end{pmatrix}. \tag{3.5}$$

If ε is small enough then $(K_\varepsilon, g_\varepsilon)$ is close to (K, g) in $\phi^{-1}W^{k+3,\infty}_\phi(g_0) \times W^{k+4,\infty}_\phi(g_0)$ thus close to (K_0, g_0) in the same space. Because of the high differentiability threshold assumed all the coefficients in the equations are in $L^\infty_{\rm loc}$, and it is easy to check that the operator P^*_ε converges to P^* when ε goes to zero in such a way that the estimates in Appendix G of [12] remain uniform. It follows that for ε small enough P^*_ε can be used in place of P^* to define the right inverse needed in the arguments of Appendix G of [12]. □

One would like to use elliptic estimate arguments to show that if (K, g) is smooth, and if $(\delta J, \delta\rho)$ is smooth, then the solution is smooth. We have not been able to implement such an argument in the spaces used above because of poor differentiability of the coefficients of the equations. This has the effect that the size of the neighborhood for which the theorem applies might depend upon k. This problem will be sidetracked by working in weighted Hölder spaces.

4. Analysis in weighted Hölder spaces

Before passing to an analysis of the regularized equation (2.16), let us show that the results established in [12] can be used to obtain a Fréchet manifold of smooth solutions of constraint equations without KIDs.

In order to obtain a coherent set-up in weighted Hölder spaces we will need to impose some more conditions on the weight functions ϕ, φ, and ψ:

1. First, note that (2.5) can be rewritten as $\phi \in C^{\ell-1}_{\phi,\phi^{-1}}$, $\psi \in C^{\ell-1}_{\phi,\psi^{-1}}$, $\varphi \in C^{\ell-1}_{\phi,\varphi^{-1}}$. When dealing with Hölder spaces one also needs to assume Hölder continuity of the derivative weights, so (renaming $\ell - 1$ to ℓ) we will assume:
$$\phi \in C^{\ell,\alpha}_{\phi,\phi^{-1}}, \qquad \psi \in C^{\ell,\alpha}_{\phi,\psi^{-1}}, \qquad \varphi \in C^{\ell,\alpha}_{\phi,\varphi^{-1}}. \tag{4.1}$$
2. As discussed in [12, Appendix B], the following conditions are useful for deriving the scaling property: Let us denote by B_p the open ball of center p with radius $\phi(p)/2$. We assume that there exist constants $C_1, C_2, C_3 > 0$ such that for all $p \in M$ and all $y \in B_p$, we have
$$C_1^{-1}\phi(p) \le \phi(y) \le C_1\phi(p), \tag{4.2}$$
$$C_2^{-1}\varphi(p) \le \varphi(y) \le C_2\varphi(p), \tag{4.3}$$
$$C_3^{-1}\psi(p) \le \psi(y) \le C_3\psi(p). \tag{4.4}$$
3. Since the tool to handle non-linearities in this paper is the inverse function theorem, we need to make sure that the changes in the initial data are small as compared to the data themselves. A necessary condition for that is that the new metric be uniformly equivalent to the original one. One way of ensuring this is
$$\psi^2\phi^2 C^{k,\alpha}_{\phi,\varphi}(g_0) \subset C^{k,\alpha}_{\phi,1}(g_0). \tag{4.5}$$
This will hold under the following condition.

Proposition 4.1. *The inequality*

$$\psi^2\phi^2\varphi^{-1} \le C \tag{4.6}$$

implies (4.5).

In order to check this the reader might wish to prove first that the conditions imposed so far imply that

Lemma 4.2. *If* $u \in C^{k,\alpha}_{\phi,\varphi_1}(g)$ *and* $v \in C^{k,\alpha}_{\phi,\varphi_2}(g)$, *with one of the* φ_a*'s satisfying* (4.3) *and* ϕ *satisfying* (4.1) *with* $\ell \ge k$, *then* $uv \in C^{k,\alpha}_{\phi,\varphi_1\varphi_2}(g)$.

Lemma 4.2 can be used to show an equivalent of Lemma 2.1 in weighted Hölder spaces.

Clearly all those conditions are fulfilled when $\phi = \varphi = \psi = 1$; they will also be fulfilled in the other examples of interest discussed in Section 6.

To proceed further some terminology will be needed.

Definition 4.3. We will say that an operator L from $\mathring{H}^3_{\phi,\psi} \times \mathring{H}^4_{\phi,\psi}$ to $\mathring{H}^1_{\phi,\psi} \times \mathring{H}^0_{\phi,\psi}$ satisfies the *weighted elliptic regularity condition* if there exists a constant C such that for all (Y, N) in $\mathring{H}^3_{\phi,\psi} \times \mathring{H}^4_{\phi,\psi}$ satisfying $L(Y, N) \in C^{k+1,\alpha}_{\phi,\varphi} \times C^{k,\alpha}_{\phi,\varphi}$ we have $(Y, N) \in C^{k+3,\alpha}_{\phi,\varphi} \times C^{k+4,\alpha}_{\phi,\varphi}$ with

$$\|(Y, N)\|_{C^{k+3,\alpha}_{\phi,\varphi}\times C^{k+4,\alpha}_{\phi,\varphi}} \le C\left(\|L(Y, N)\|_{C^{k+1,\alpha}_{\phi,\varphi}\times C^{k,\alpha}_{\phi,\varphi}} + \|(Y, N)\|_{H^3_{\phi,\psi}\times H^4_{\phi,\psi}}\right). \tag{4.7}$$

When $L_{\phi,\psi}$ defined in (2.10) satisfies the weighted elliptic regularity condition one has the following proposition.

Proposition 4.4 (Proposition 3.16 of [12]). *Let* $k \in \mathbb{N}$, $0 < \alpha < 1$, *assume that* (4.1) *with* $\ell \ge k + 4$ *holds, and that* (4.2)–(4.4) *and* (4.6) *hold. In addition to the hypotheses of* Theorem 3.1 *with* $\varepsilon = 0$, *suppose that* $g_0 \in C^{k+4,\alpha}$, *and that*

$$\mathrm{Ric}(g_0) \in \phi^{-2}C^{k+2,\alpha}_{\phi,1}(g_0), \qquad K_0 \in \phi^{-1}C^{k+3,\alpha}_{\phi,1}(g_0).$$

We further assume that the weights ϕ, φ *and* ψ *have the* scaling property. *Suppose, next, that we have the continuous inclusions*

$$\psi^2\phi^2 C^{i,\alpha}_{\phi,\varphi^2}(g) \subset \mathring{H}^i_{\phi,\psi}(g) \tag{4.8}$$

for $i = k, k+1$, *with the inclusion norms uniformly bounded for g close to* g_0 *in* $C^{k+4,\alpha}_{\phi,1}(g_0)$. *Assume finally that* $L_{\phi,\psi}(K, g)$ *satisfies the weighted elliptic regularity condition, with a uniform constant C in* (4.7) *for* (K, g) *close to* (K_0, g_0) *in* $\phi^{-1}C^{k+3,\alpha}_{\phi,1}(g_0) \times C^{k+4,\alpha}_{\phi,1}(g_0)$. *If the source* $(\delta J, \delta\rho)$ *is in* $\psi^2(\mathring{H}^1_{\phi,\psi}(g) \times \mathring{H}^0_{\phi,\psi}(g)) \cap \psi^2(C^{k+1,\alpha}_{\phi,\varphi}(g) \times C^{k,\alpha}_{\phi,\varphi}(g))$, *with sufficiently small norm, then the solution obtained in* Theorem 3.1 *with* $\varepsilon = 0$ *is in*

$$\psi^2(\phi\mathring{H}^2_{\phi,\psi}(g) \times \phi^2\mathring{H}^2_{\phi,\psi}(g)) \cap \psi^2(\phi C^{k+2,\alpha}_{\phi,\varphi}(g) \times \phi^2 C^{k+2,\alpha}_{\phi,\varphi}(g)).$$

Proposition 4.4 gives existence of Hölder continuous solutions. We can apply the usual bootstrap arguments to those solutions to obtain smoothness, when all the objects at hand are smooth (compare the proof of Theorem 4.9).

Proposition 4.5 (Proposition 3.17 of [12]). *Let $k \in \mathbb{N}$, $\alpha \in (0,1)$, assume that (4.1) with $\ell \geq k+4$ holds, and that (4.2)–(4.4) and (4.6) hold. Suppose moreover that the scaling property holds. Assume that $(K, g) \in C^{k+3,\alpha} \times C^{k+4,\alpha}$ and $(Y, N) \in C^{3,\alpha}_{\phi,\varphi}(g) \times C^{4,\alpha}_{\phi,\varphi}(g)$. If*

$$\begin{pmatrix} J \\ \rho \end{pmatrix} [(K,g) + \psi^2 \Phi^2 P^*_{(K,g)}(Y,N)] - \begin{pmatrix} J \\ \rho \end{pmatrix} [(K,g)] \in \psi^2 (C^{k+1,\alpha}_{\phi,\varphi}(g) \times C^{k,\alpha}_{\phi,\varphi}(g)), \tag{4.9}$$

then $(Y, N) \in C^{k+3,\alpha}_{\phi,\varphi}(g) \times C^{k+4,\alpha}_{\phi,\varphi}(g))$, thus

$$(\delta K, \delta g) \in \psi^2 (\phi C^{k+2,\alpha}_{\phi,\varphi}(g) \times \phi^2 C^{k+2,\alpha}_{\phi,\varphi}(g)). \tag{4.10}$$

Example 2.4 (continued). Applying the last two propositions to the setup of Example 2.4 one finds that smooth solutions of the linearized constraint equations correspond to smooth solutions of the full non-linear constraint equations. This leads then to a Fréchet manifold of smooth solutions near every smooth solution. The argument at the end of proof of Theorem 5.2 justifies the isomorphism property on the overlaps of the coordinate charts, and we have therefore obtained the Fischer–Marsden–Moncrief result.

Theorem 4.6 (Fischer et al. [21]). *Let M be a compact manifold with boundary. Then the level sets $\mathcal{S}$ of the constraints map form a submanifold of the set of smooth (K, g)'s at all (K, g) which have no KIDs. Each connected component $\mathcal{S}_0$ thereof is a Fréchet manifold modeled on* Ker $P_{(K,g)} \subset C^\infty \times C^\infty$, *where (K, g) is an arbitrary element of $\mathcal{S}_0$.*

Specializing all the considerations so far to the case $K \equiv Y \equiv 0$ one recovers a theorem essentially due to Fischer and Marsden.

Theorem 4.7 (Fischer and Marsden [19]). *Let M be a compact manifold with boundary. Then the level sets of the scalar curvature functional on the space of smooth metrics form a submanifold at all g which do not correspond to the space-part of some static solution of the vacuum Einstein equations with a cosmological constant. Each connected component $\mathcal{S}_0$ thereof is a Fréchet manifold modeled on the kernel of DR, as calculated at some arbitrarily chosen metric $g \in \mathcal{S}_0$.*

The argument of the proof of Theorem 3.1 also establishes the following proposition.

Proposition 4.8. *Under the conditions of Proposition 4.4, assume that (3.1) holds, and suppose that $\psi^{-2} P \psi^2 \Phi^2 P^*_\varepsilon$ satisfies the weighted elliptic regularity condition, with the constant C in (4.7) being uniform for (K, g) close to (K_0, g_0) in $\phi^{-1} C^{k_0+3,\alpha}_{\phi,1}(g_0) \times C^{k_0+4,\alpha}_{\phi,1}(g_0)$ and ε small enough. Then Proposition 4.4 remains valid with k replaced by k_0 and P^* replaced by P^*_ε when ε is small enough.*

Our first main result is the equivalent of Proposition 4.5 with less regularity conditions on (K, g).

Theorem 4.9. *Let $k \in \mathbb{N}$, $k \geq 2$, $\alpha \in (0,1)$, assume that (4.1) with $\ell \geq k+4$ holds, and that (4.2)–(4.4) and (4.6) hold. Under (3.1), suppose that the scaling condition holds for $\varepsilon > 0$. Assume that $(K, g) \in C^{k+2,\alpha} \times C^{k+2,\alpha}$ and $(Y, N) \in C^{3,\alpha}_{\phi,\varphi}(g) \times C^{4,\alpha}_{\phi,\varphi}(g)$. For ε small, if*

$$\begin{pmatrix} J \\ \rho \end{pmatrix} [(K, g) + \psi^2 \Phi^2 P^*_\varepsilon(Y, N)] - \begin{pmatrix} J \\ \rho \end{pmatrix} [(K, g)] \in \psi^2 (C^{k+1,\alpha}_{\phi,\varphi}(g) \times C^{k,\alpha}_{\phi,\varphi}(g)), \tag{4.11}$$

then $(Y, N) \in C^{k+3,\alpha}_{\phi,\varphi}(g_\varepsilon) \times C^{k+4,\alpha}_{\phi,\varphi}(g_\varepsilon)$, thus

$$(\delta K, \delta g) \in \psi^2 (\phi C^{k+2,\alpha}_{\phi,\varphi}(g) \times \phi^2 C^{k+2,\alpha}_{\phi,\varphi}(g)).$$

Remark 4.10. The appearance of ε in the claim that $(Y, N) \in C^{k+3,\alpha}_{\phi,\varphi}(g_\varepsilon) \times C^{k+4,\alpha}_{\phi,\varphi}(g_\varepsilon)$ is due to the fact that g is a priori not sufficiently differentiable to be able to define spaces such as $C^{k+3,\alpha}_{\phi,\varphi}(g_\varepsilon)$. Any fixed metric uniformly equivalent to g, with appropriate weighted differentiability properties, could be used instead of g_ϵ in the definition of those spaces.

Proof. It suffices to rewrite the rescaled non-linear elliptic equation (4.9) for (Y, N) as a linear elliptic equation for (Y, N) and freeze coefficients (depending on $(K + \delta K, g + \delta g)$ hence on (Y, N)). The interior Hölder estimates [28, Theorem 6.2.5, p. 223] on the sets $\hat{\Omega}_\alpha$ appearing in the definition of scaling property give the local regularity, and the scaling property gives the global weighted regularity. □

5. Banach manifold structure

Throughout this section the symbol g_0 denotes a fixed metric with (local) regularity $C^{m+4,\alpha}$ on M.

For $k, l \in \{0, \ldots, m+4\}$, $\alpha \in (0,1)$ and g a metric in $C^{\max(l,k),\alpha}$, we define the Banach space

$$\Lambda^{l,k,\alpha}_{\phi,\psi,\varphi}(g) = \mathring{H}^l_{\phi,\psi}(g) \cap C^{k,\alpha}_{\phi,\varphi}(g),$$

equipped with a norm being the sum of the two norms. (It should be clear from (4.10) that this is the topology which one needs to use on the space of the metrics when using the methods described above.) When $k \in \{0, \ldots, m+2\}$ and when (4.5) hold, we define the following *open* subset of $\psi^2\phi^2\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0)$ of symmetric two-covariant tensor fields on M:

$$\mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0) := \Big\{ h \in \psi^2\phi^2\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0), \quad g_0 + h \text{ is a metric uniformly equivalent to } g_0 \Big\}.$$

We note the continuous inclusions:

$$\mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \subset \psi^2\phi^2\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \subset C^{k+2,\alpha}_{\phi,1}(g_0)(\subset W^{k+2,\infty}_{\phi}(g_0)). \tag{5.1}$$

We have the following lemma.

Lemma 5.1. *Let $k \in \{0,\dots,m+2\}$ and $l \in \{0,\dots,k+2\}$. Assume (4.1)–(4.5). Then for all $h \in \mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0)$ we have*

$$C^{l,\alpha}_{\phi,\varphi}(g_0+h) = C^{l,\alpha}_{\phi,\varphi}(g_0), \qquad \mathring{H}^{l}_{\phi,\psi}(g_0+h) = \mathring{H}^{l}_{\phi,\psi}(g_0)$$

with equivalent norms. In particular,

$$\mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0+h) = \mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0).$$

Proof. Let $g = g_0 + h$, we define $T = \Gamma - \Gamma_0$, since g is uniformly equivalent to g_0 the usual formula for T allows one to estimate this by $C|\nabla_0 h|_{g_0}$. By the middle inclusion in (5.1) we then have $T \in \phi^{-1}C^{k+1,\alpha}_{\phi,1}(g_0)$. For a tensor u, we have $\|u\|_{g_0}$ uniformly equivalent to $\|u\|_g$. For the derivatives we write

$$\nabla u = \nabla_0 u + (\nabla - \nabla_0)u = \nabla_0 u - Tu.$$

If $u \in C^{l,\alpha}_{\phi,\varphi}(g_0)$ by the product Lemma 4.2 we obtain $\phi\nabla u \in C^{0,\alpha}_{\phi,\varphi}(g)$. The higher derivatives follow by induction. This shows that $C^{l,\alpha}_{\phi,\varphi}(g_0) \subset C^{l,\alpha}_{\phi,\varphi}(g)$.

We note that the above implies that $-h \in \psi^2\phi^2 C^{k+2,\alpha}_{\phi,\varphi}(g)$, and the reverse inclusion follows by symmetry.

The proof for the Sobolev spaces is identical. □

Theorem 5.2. *Let $k \in \{2,\dots,m\}$ and $\alpha \in (0,1)$, and*

$$\mathrm{Ric}(g_0) \in \phi^{-2}C^{k,\alpha}_{\phi,1}(g_0), \qquad K_0 \in \phi^{-1}C^{k+2,\alpha}_{\phi,1}(g_0). \tag{5.2}$$

Suppose that the scaling property and the weighted regularity condition hold, and that (4.1)–(4.4) together with (4.6) are satisfied. Assume also that for all[10]

$$(Q,h) \in \psi^2\phi\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \times \mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \tag{5.3}$$

there exists a compact set $\mathscr{K} \subset M$ such that for all $\mathring{H}^1_{\phi,\psi}(g_0)$ vector fields Y and $\mathring{H}^2_{\phi,\psi}(g_0)$ functions N, both supported in $M \setminus \mathscr{K}$, the inequality (2.12) holds with (K_0, g_0) there replaced by (K_0+Q, g_0+h). Suppose, further, that for all (Q,h) as in (5.3) the map

$$(\delta K, \delta g) \to (J,\rho)(K_0+Q+\delta K, g_0+h+\delta g) - (J,\rho)(K_0+Q, g_0+h) \tag{5.4}$$

is differentiable from a neighborhood of zero in $\psi^2\phi\mathring{H}^2_{\phi,\psi}(g_0) \times \psi^2\phi^2\mathring{H}^2_{\phi,\psi}(g_0)$ to $\psi^2\mathring{H}^1_{\phi,\psi}(g_0) \times \psi^2\mathring{H}^0_{\phi,\psi}(g_0)$. Consider any non-empty connected component of the set of KID-free level-sets:

[10] Actually it suffices to assume that this hypothesis holds on $\mathcal{S}_{J_0,\rho_0}$.

$$\mathcal{S}_{J_0,\rho_0} = \Big\{(Q,h) \in \psi^2\phi\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \times \mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0), \\ (J,\rho)(K_0+Q, g_0+h) = (J_0,\rho_0), \operatorname{Ker} P^*_{K_0+Q,g_0+h} = \{0\}\Big\}. \tag{5.5}$$

If there exists a smoothing operation (3.1) with $k_0 = 0$, then $\mathcal{S}_{J_0,\rho_0}$ is an embedded submanifold of $\psi^2\phi\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \times \mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0)$.

Remark 5.3. Differentiability of the map (5.4) in weighted Sobolev spaces typically requires $k > n/2$, this can be actually avoided by requiring instead differentiability of (5.4) as a map from $\psi^2\phi\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \times \psi^2\phi^2\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0)$ to $\psi^2\Lambda^{1,k+1,\alpha}_{\phi,\psi,\varphi}(g_0) \times \psi^2\Lambda^{0,k,\alpha}_{\phi,\psi,\varphi}(g_0)$.

Remark 5.4. We note that a necessary condition for $\mathcal{S}_{J_0,\rho_0} \neq \emptyset$ is

$$J_0 \in \phi^{-2}C^{k+1,\alpha}_{\phi,1}(g_0), \qquad \rho_0 \in \phi^{-2}C^{k,\alpha}_{\phi,1}(g_0). \tag{5.6}$$

In any case it seems that the situation of main interest is $J_0 = \rho_0 = 0$.

Remark 5.5. The kernel in (5.5) is that of the operator $P^*_{K_0+Q,g_0+h}$ acting from $H^1_{\phi,\psi} \times H^2_{\phi,\psi}$ to $\phi^{-1}H^0_{\phi,\psi} \times \phi^{-2}H^0_{\phi,\psi}$. We note that elliptic regularity shows that elements of the kernel are as differentiable as the metric allows, so the elements of the kernel are continuously differentiable solutions satisfying appropriate asymptotic properties.

Remark 5.6. We do not assume that $(J_0,\rho_0) = (J,\rho)(K_0,\rho_0)$. Even if this last equality holds, $(0,0)$ will fail to be in $\mathcal{S}_{J_0,\rho_0}$ if there are KIDs at (K_0, g_0).

Remark 5.7. We do not assume (Q,h) to be small.

Remark 5.8. Some rather general conditions which guarantee existence of smoothing operators (3.1) are given in Appendix B.

Proof. We wish to apply Lemma A.1 with $x = (\delta K, \delta g)$ and

$$u = Du(0) = \psi^2\Phi^2 P^*_{\varepsilon,(K_0+Q_0,g_0+h_0)},$$

$$v(x) = \psi^{-2}[(J,\rho)(K_0+Q_0+\delta K, g_0+h_0+\delta g) - (J,\rho)(K_0+Q_0, g_0+h_0)],$$

$$Dv(0) = \psi^{-2}P_{K_0+Q_0,g_0+h_0},$$

$$E = \Lambda^{3,k+3,\alpha}_{\phi,\psi,\varphi}(g_0) \times \Lambda^{4,k+4,\alpha}_{\phi,\psi,\varphi}(g_0),$$

$$F = \psi^2\Phi(\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \times \Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi})(g_0),$$

$$G = \Lambda^{1,k+1,\alpha}_{\phi,\psi,\varphi}(g_0) \times \Lambda^{0,k,\alpha}_{\phi,\psi,\varphi}(g_0).$$

We start by verifying that

$$L_\varepsilon := \psi^{-2}P_{K_0+Q_0,g_0+h_0}\psi^2\Phi^2 P^*_{\varepsilon,(K_0+Q_0,g_0+h_0)}$$

is an isomorphism from E to G. We wish to use Theorem 2.2 with (K_0, g_0) there replaced with (K_0+Q_0, g_0+h_0), and with k there equal zero; the needed regularity conditions on

$\mathrm{Ric}(g_0+h_0)$ can be established by the calculations of Lemma 5.1 using (5.2) (recall that $k \geq 2$), while the condition on K_0+Q_0 follows immediately from (5.2). The remaining conditions are satisfied by hypothesis. Since there are no KIDs we conclude that $L_{\varepsilon=0} \equiv L_{\phi,\psi}$ (see (2.10)) is an isomorphism from $\mathring{H}^3_{\phi,\psi}(g_0+h_0) \times \mathring{H}^4_{\phi,\psi}(g_0+h_0)$ to $\mathring{H}^1_{\phi,\psi}(g_0+h_0) \times \mathring{H}^0_{\phi,\psi}(g_0+h_0)$; those spaces coincide with the ones based on g_0 by Lemma 5.1. The hypothesis of the existence of the smoothing operation (3.1) with k there equal 0 implies that $P^*_\varepsilon \to_{\varepsilon\to 0} P$ in the space of linear maps from $\mathring{H}^3_{\phi,\psi} \times \mathring{H}^4_{\phi,\psi}$ to $\phi^{-1}\mathring{H}^2_{\phi,\psi} \times \phi^{-2}\mathring{H}^2_{\phi,\psi}$. It follows that L_ε is an isomorphism from $\mathring{H}^3_{\phi,\psi} \times \mathring{H}^4_{\phi,\psi}$ to $\mathring{H}^1_{\phi,\psi} \times \mathring{H}^0_{\phi,\psi}$ for ε small enough. So L_ε is injective on E. The weighted elliptic regularity condition implies Hölder regularity of the solution, and surjectivity follows.

We leave it as an exercise to the reader to prove the following lemma using Lemma 4.2 together with the arguments in Lemma 5.1.

Lemma 5.9. *Under the conditions of* Theorem 5.2, *the map* (5.4) *is smooth from a neighborhood of zero in* $\psi^2\phi C^{k+2,\alpha}_{\phi,\varphi}(g_0) \times \psi^2\phi^2 C^{k+2,\alpha}_{\phi,\varphi}(g_0)$ *to* $\psi^2 C^{k+1,\alpha}_{\phi,\varphi}(g_0) \times \psi^2 C^{k,\alpha}_{\phi,\varphi}(g_0)$.

This does not suffice to prove differentiability of v, because of the topology involved; however, differentiability with respect to the Sobolev topology holds by hypothesis.

It follows that near (Q_0, h_0) the set $\mathcal{S}_{J_0,\rho_0}$ is an embedded submanifold modeled on the kernel of $P_{K_0+Q_0,g_0+h_0}$.

Remark 5.10. The proof above actually gives a foliation of a neighborhood of (Q_0, h_0) in $F = \psi^2\Phi(\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \times \Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0))$. Indeed, under the conditions of Theorem 5.2 we can use Lemma A.2 from Appendix A with the same spaces as those in Theorem 5.2 and with $L = u$. For all $(Q_0, h_0) \in \mathcal{S}_{J_0,\rho_0}$, there is a neighborhood V of zero in $G = \Lambda^{1,k+1,\alpha}_{\phi,\psi,\varphi}(g_0) \times \Lambda^{0,k,\alpha}_{\phi,\psi,\varphi}(g_0)$, such that

$$\Big\{(Q,h) \in \psi^2\phi\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \times \mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0),\\ (J,\rho)(K_0+Q, g_0+h) = (J_0,\rho_0) + (\delta J, \delta\rho), \mathrm{Ker}\, P^*_{K_0+Q,g_0+h} = \{0\}\Big\}_{(\delta J,\delta\rho)\in V}$$

is a foliation. As we can do that for all points $(Q_0, h_0) \in \mathcal{S}_{J_0,\rho_0}$, we obtain a foliation of a neighborhood of $\mathcal{S}_{(J_0,\rho_0)}$ in F.

In fact, if we denote by F_0 the open subset of $\psi^2\phi\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}(g_0) \times \mathcal{A}^{k+2,\alpha}_{\phi,\psi,\varphi}(g_0)$ of elements (Q, h) such that

$$\mathrm{Ker}\, P^*_{K_0+Q,g_0+h} = \{0\},$$

then the map from F_0 to G defined by

$$f(Q,h) = (J,\rho)(K_0+Q, g_0+h) - (J,\rho)(K_0, g_0)$$

is a submersion. In particular the levels sets

$$\{(Q,h) \in F_0, \quad f(Q,h) = (\delta J, \delta\rho)\}_{(\delta J,\delta\rho)\in G},$$

provide a foliation of F_0.

For completeness we note the following result.

Corollary 5.11. *Every non-empty connected component of $\mathcal{S}_{J_0,\rho_0}$ defined in (5.5) is a Banach manifold modeled on the kernel of $P_{K_0+Q_0,g_0+h_0}$ in $\psi^2\Phi(\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi}\times\Lambda^{2,k+2,\alpha}_{\phi,\psi,\varphi})$ for an (arbitrarily chosen) $(Q_0,h_0)\in\mathcal{S}_{J_0,\rho_0}$.*

Proof. An embedded submanifold of a Hausdorff space is necessarily Hausdorff. The local coordinate patches are provided by the maps which model $\mathcal{S}$ on its tangent spaces constructed in the proof of Theorem 5.2. It remains to show that all the kernels are (Q,h)-independent. The proof of Theorem 5.2 shows that for all $(Q,h)\in\mathcal{S}_{J_0,\rho_0}$ there exists a neighborhood thereof in $\mathcal{S}_{J_0,\rho_0}$ which is arc-connected. It then follows that a connected component of $\mathcal{S}_{J_0,\rho_0}$ is in fact arc-connected, by observing that the set of metrics in $\mathcal{S}_{J_0,\rho_0}$ which can be connected to a fixed metric in $\mathcal{S}_{J_0,\rho_0}$ by a continuous curve contained in $\mathcal{S}_{J_0,\rho_0}$ is open and closed in $\mathcal{S}_{J_0,\rho_0}$. Thus for two couples $(Q_0,h_0)\in\mathcal{S}_{J_0,\rho_0}$ and $(Q_p,h_p)\in\mathcal{S}_{J_0,\rho_0}$, there exists an arc γ in $\mathcal{S}_{J_0,\rho_0}$ from one to the other. For each point $x=(Q,h)\in\gamma$, there exists an $r_x>0$ such that $\mathcal{S}_{J_0,\rho_0}\cap B_F(x,r_x)$ is diffeomorphic to an open subset of the kernel of $P_{(K+Q,g+h)}$. As γ is compact, there exists a finite number of points $\{x_i=(Q_i,h_i)\in\gamma, i=0,\dots,p\}$ such that the union of the $B_F(x_i,r_{x_i})$'s covers γ. This provides a finite chain of diffeomorphisms, a composition of which identifies the kernel of $P_{(K+Q_0,g+h_0)}$ with that of $P_{(K+Q_p,g+h_p)}$. □

6. Applications

6.1. Compact manifolds without boundary

In this section we apply Theorem 5.2 to the case where M is a compact manifold without boundary. As already pointed out, in this case we take

$$\phi=\psi=\varphi=1,$$

then the spaces we work with are the standard (non-weighted) Sobolev and Hölder spaces.

Theorem 6.1. *Let $k\geq 2$ and $\alpha\in(0,1)$. Let $g_0\in C^{k+2,\alpha}$, $K_0\in C^{k+2,\alpha}$. Then any non-empty connected component of the set*

$$\mathcal{S}_{J_0,\rho_0}=\Big\{(Q,h)\in C^{k+2,\alpha}\times\mathcal{A}^{k+2,\alpha}_{1,1,1},\ (J,\rho)(K_0+Q,g_0+h)=(J_0,\rho_0),\ \mathrm{Ker}\,P^*_{K_0+Q,g_0+h}=\{0\}\Big\}$$

is an embedded submanifold of $C^{k+2,\alpha}\times C^{k+2,\alpha}$.

In Theorem 6.1 the kernel of the operator $P^*_{K_0+Q,g_0+h}$ can be taken as that of an operator from $H^1\times H^2$ to $H^0\times H^0$, or from $C^1\times C^2$ to $C^0\times C^0$, or from $C^{k+1,\alpha}\times C^{k+2,\alpha}$ to $C^{k,\alpha}\times C^{k,\alpha}$.

6.2. Asymptotically flat manifolds without boundary

In this section we apply Theorem 5.2 to the case where M is an asymptotically flat manifold without boundary and with compact interior; by definition, this means that M is the

union of a compact set with a finite number of regions, called ends, which are diffeomorphic to $\mathbb{R}^n \setminus B(0,R)$ for some R. We denote by $\hat\delta$ some arbitrarily chosen metric of $C^{m+4,\alpha}$ differentiability class which coincides with the Euclidean one in the asymptotically flat regions. Here and in the following sections the index m corresponds to the differentiability of the background, typically $m=\infty$ will be appropriate, however in some situations it might be useful to have a background with finite differentiability. The weight functions are $\phi = r$, with φ and ψ—powers of r in the asymptotic regions, which are easily seen to satisfy (4.1)–(4.4) in the asymptotic regions. We extend the function r to a smooth strictly positive function in the compact region, then the requirements on the weight functions are satisfied globally. It is convenient to relabel the $\mathring{H}^k_{r,r^\alpha}$ and the $C^{k,\alpha}_{r,r^s}$ spaces as follows: choose some $m \in \mathbb{N}$, for $k \in \{0,\dots,m+4\}$, $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$ we set

$$\mathring{\mathscr{H}}^\beta_k = \mathring{H}^k_{r,r^{-n/2-\beta}}(\hat\delta), \qquad C^\beta_{k,\alpha} = C^{k,\alpha}_{r,r^{-\beta}}(\hat\delta).$$

For $\lambda \in \mathbb{R}$ we define

$$\Lambda^\lambda_{k,\alpha} = \Lambda^{2,k,\alpha}_{r,r^{-n/2-\lambda},r^{-\lambda}} = \mathring{\mathscr{H}}^\lambda_2 \cap C^\lambda_{k,\alpha}. \tag{6.1}$$

We also define for $k \geq 2$:

$$\begin{aligned}\mathcal{M}_{k,\alpha} = \{&g \text{ is a metric uniformly equivalent to } \hat\delta,\\ &g-\hat\delta \in C^0_{k,\alpha}, |\nabla^{(l)}_{\hat\delta}(g-\hat\delta)|_{\hat\delta} = o(r^{-l}),\ \ 0 \leq l \leq 2\}.\end{aligned} \tag{6.2}$$

For $\gamma \geq 0$ and $k \leq m+2$ we let

$$\mathcal{A}^{-\gamma}_{k+2,\alpha} = \mathcal{A}^{2,k+2,\alpha}_{r,r^{n/2-2-\gamma},r^{-\gamma-2+n}}(\hat\delta) \subset C^{-\gamma}_{k+2,\alpha} \subset C^0_{k+2,\alpha}.$$

This corresponds to $\psi = r^{n/2-2-\gamma}$, $\varphi = r^{-\gamma-2+n}$, and since $\phi = r$ the condition (4.6) holds precisely for $\gamma \geq 0$. The choice of weights here is justified by Theorem 7.7 in [12].

With the labeling above, a metric such that $g-\hat\delta \in \mathcal{A}^\beta_{k+2,\alpha}$ with $\beta \leq 0$ clearly differs from the Euclidean metric by $O(r^\beta)$. In order to see that this is actually $O(r^\beta)$, for $i \in \mathbb{N}$ large set

$$\Gamma_i = \Gamma(i,i+1) = \overline{B(0,i+1)} \setminus \overline{B(0,i)}.$$

Let $f \in \mathring{\mathscr{H}}^\beta_0 \cap C^\beta_1$ and let $x_i \in \bar\Gamma_i$ be any point such that

$$|f(x_i)| = \sup_{y\in\Gamma_i} |f(y)|.$$

If $\|f\|_{C^\beta_1} = 0$ there is nothing to prove, otherwise let $C = (7/8)^{\beta-1}$, set

$$r_i = \min\left(\frac{1}{8}, \frac{|f(x_i)|i^{-\beta}}{2C\|f\|_{C^\beta_1}}\right) i.$$

For $y \in B(x_i,r_i)$ we have

$$|f(y)-f(x_i)| \leq \left(\sup_{z\in B(x_i,r_i)} |Df|(z)\right)|y-x_i| \leq C\|f\|_{C^\beta_1} i^{\beta-1}|y-x_i| \leq \frac{|f(x_i)|}{2},$$

which implies

$$\int_{B(x_i,r_i)} f^2(y)|y|^{-2\beta-n}d^n y \geq C'\left(\frac{f(x_i)}{2i^\beta}\right)^2\left(\frac{r_i}{i}\right)^n.$$

The left-hand-side goes to zero as i goes to infinity by the dominated convergence theorem, which easily implies the result.

The reference metric g_0 will be taken to be such that $g_0 \in \mathcal{M}_{k+2,\alpha}$. The reference K_0 can be taken to be zero, but any $K_0 \in C^{-1}_{k+2,\alpha}$ with $K_0 = o(r^{-1})$ will do; this last condition ensures that the $\mathscr{K}$-set condition of Theorem 5.2 holds, see [12, Section 7] for details.

We take the smoothing operation to be the one in Appendix B.1, the only thing which needs to be checked is the uniform covering condition (B.3): We write

$$\mathbb{R}^n \setminus \overline{B(0,R)} = \bigcup_{i=0}^{\infty} \Gamma(2^i R, 2^{i+1}R).$$

Now $\Gamma(1,2)$ can be covered by a finite number N of ball's $B(x_j, |x_j|/8)$ with $x_j \in \Gamma(1,2)$. Then $\Gamma(2^iR, 2^{i+1}R)$ can be covered by N balls $B(2^iRx_j, 2^{i-3}R|x_j|)$ with $2^iRx_j \in \Gamma(2^iR, 2^{i+1}R)$. It is then clear that $\mathbb{R}^n \setminus \bar{B}(0,R)$ can be covered by a countable set of balls $B(y_k, |y_k|/8)$ with the property that for all $k \in \mathbb{N}$,

$$\#\{l, \quad B(y_l, \tfrac{1}{2}|y_l|) \cap B(y_k, \tfrac{1}{2}|y_k|) \neq \emptyset\} \leq 3N,$$

as desired.

We note that the differentiability of the map (5.4) follows from the weighted equivalent of the Schauder ring property of $H^k \cap L^\infty$. All the remaining hypotheses of Theorem 5.2 will be satisfied by [12] (compare Section 7 there) under the following conditions.

Theorem 6.2. *Let $m \in \mathbb{N}$, $k \in \{2, \ldots, m\}$, $\alpha \in (0,1)$, $g_0 \in \mathcal{M}_{k+2,\alpha}$, $K_0 \in C^{-1}_{k+2,\alpha}$ with $K_0 = o(r^{-1})$. Let $\beta \geq 0$, $\beta \notin \{n-2, n-1\}$. Then any non-empty connected component of the set of KID-free initial data*:

$$\mathcal{S}_{J_0,\rho_0} = \Big\{(Q,h) \in \Lambda^{-\beta-1}_{k+2,\alpha} \times \mathcal{A}^{-\beta}_{k+2,\alpha}, \\ (J,\rho)(K_0+Q, g_0+h) = (J_0,\rho_0), \operatorname{Ker} P^*_{K_0+Q,g_0+h} = \{0\}\Big\}$$

is an embedded submanifold of $\Lambda^{-\beta-1}_{k+2,\alpha} \times \mathcal{A}^{-\beta}_{k+2,\alpha}$.

In the above the kernel of the operator $P^*_{K_0+Q,g_0+h}$ is viewed as that of an operator from $\mathring{\mathscr{H}}^{\beta+2-n}_1 \times \mathring{\mathscr{H}}^{\beta+2-n}_2$ to $\mathring{\mathscr{H}}^{\beta-n+1}_0 \times \mathring{\mathscr{H}}^{\beta-n}_0$. Elliptic regularity implies that elements of this kernel are classically differentiable KIDs such that $Y = o(r^{\beta+2-n})$, $N = o(r^{\beta+2-n})$. It is known that for $0 < \beta < n-2$ there are no non-trivial such KIDs, so we obtain the following corollary.

Corollary 6.3. *Under the conditions of Theorem 6.2, if $\beta \in (0, n-2)$ then all the level sets of the constraints map*

$$\left\{(Q,h)\in \Lambda^{-t+1}_{k+2,\alpha}\times \mathcal{A}^{-t+2}_{k+2,\alpha},\ (J,\rho)(K_0+Q,g_0+h)=(J_0,\rho_0)\right\}$$

are embedded submanifolds of $\Lambda^{-t+1}_{k+2,\alpha}\times \mathcal{A}^{-t+2}_{k+2,\alpha}$.

Thus, within the above set of weights the set of solutions of the vacuum constraint equations does not have any manifold singularities. On the other hand, such singularities will occur at solutions with KIDs if higher values of β are used. The interest of such higher β's relies in the fact that the resulting manifolds of solutions possess fixed energy–momentum, or angular momentum, or higher multipoles, depending upon the value of β.

6.3. Compact manifold with boundary

In this section we apply Theorem 5.2 to the case where $\bar{M}$ is a compact manifold with smooth boundary ∂M. We wish to construct manifolds of initial data on $\bar{M}$ with prescribed boundary values on ∂M, as well as a prescribed number of transverse derivatives at the boundary. Let γ be any fixed auxiliary Riemannian metric of $C^{m+4,\alpha}(\bar{M})$ differentiability class. Let $x\geq 0$ be a function that vanishes precisely on ∂M, with dx nowhere vanishing there. We start by considering power-law weighted spaces defined as

$$\mathring{\mathsf{H}}^s_k=\mathring{H}^k_{1,x^{-s-n/2}}(\gamma),\qquad \mathsf{C}^s_{k,\alpha}=C^{k,\alpha}_{x,x^{-s}}(\gamma),\qquad \Lambda^s_{k,\alpha}=\Lambda^{2,k,\alpha}_{x,x^{-s-n/2},x^{-s}}=\mathring{\mathsf{H}}^s_2\cap \mathsf{C}^s_{k,\alpha}.$$

We also define for $m+4\geq k\geq 2$:

$$\mathcal{M}_{k,\alpha}=\Big\{g \text{ metric uniformly equivalent to } \gamma,\ g-\gamma\in \mathsf{C}^0_{k,\alpha},\ |\nabla^{(l)}_\gamma(g-\gamma)|_\gamma=o(x^{-l}),\ \ 0\leq l\leq 2\Big\} \tag{6.3}$$

(this differs from (6.2) by a different background metric and different functional spaces, we hope that this ambiguity will not lead to confusions). For $\sigma\geq 0$ and $m+2\geq k\geq 0$ we set

$$\mathcal{A}^\sigma_{k+2,\alpha}=\mathcal{A}^{k+2,\alpha}_{x,x^{\sigma-2+n/2},x^{\sigma-2+n}}(\gamma)\subset \mathsf{C}^\sigma_{k+2,\alpha}\subset \mathsf{C}^0_{k+2,\alpha}.$$

The first inclusion shows that metrics of the form $g=\gamma+h$ with $h\in\mathcal{A}^\sigma_{k+2,\alpha}$ approach γ at ∂M at least as $O(x^\sigma)$, and in fact an argument similar to the one in Section 6.2 shows that this is actually at least $o(x^\sigma)$. The above corresponds to $\phi=x$, $\varphi=x^{\sigma+n-2}$, $\psi=x^{\sigma+n/2-2}$, with the choices being justified as follows: In [12, Theorem 5.6] we obtain metrics such that δg is in $\mathring{H}_{x,x^{-(s-n+2)-n/2}}\cap C_{x,x^{-(s-n+2)}}$. It is convenient to number the spaces according to the decay rate of δg near the boundary, so we set $\sigma=s-n+2$. In the current set-up we have δg in $\psi^2\phi^2(\mathring{H}_{\phi,\psi}\cap C_{\phi,\varphi})$, which after straightforward algebra uniquely leads to the weights above.

In order to obtain the required smoothing operator we use the results in Appendix B.1, we need to justify the covering condition there. Let $T>0$, we set

$$\mathbb{R}^n_+=\{x=(x^1,\dots,x^n)\in\mathbb{R}^n,\ x^n>0\}=\bigcup_{i=-\infty}^{\infty}\mathcal{B}(2^iT,2^{1+i}T),$$

where $\mathcal{B}(2^iT,2^{i+1}T)=\{x=(x^1,\dots,x^n)\in\mathbb{R}^n,\ 2^iT<x^n\leq 2^{i+1}T\}$. We have $\partial\mathbb{R}^n_+=\{x\in\mathbb{R}^n,\ \ x^n=0\}$. $\mathcal{B}(1,2)$ can be covered by closed cubes with edge sizes one, with

pairwise intersections empty, or along the faces of the cubes. We choose one of those cubes, call it K_1, and we cover it by $N(n)$ balls $B(x_i, |x_i|/8)$ with $x_i \in K_1$; every other cube in $\mathcal{B}(1,2)$ will then be covered by $N(n)$ balls by translating the balls covering K_1. We cover $\mathcal{B}(1/2,1)$ with cubes of edge sizes 1/2, intersecting along the faces only, such that a cube of $\mathcal{B}(1,2)$ intersects precisely 2^{n-1} cubes of $\mathcal{B}(1/2,1)$. Each cube of $\mathcal{B}(1/2,1)$ can be covered by $N(n)$ balls $B(x_i, |x_i|/8)$ with x_i in that cube by scaling by 1/2 and translating the balls covering K_1. An inductive repetition of the procedure leads to a covering of $\mathbb{R}^n_+$ by a countable set of balls $B(y_k, |y_k|/8)$ with the property that for all k,

$$\#\{l, \quad B(y_l, \tfrac{1}{2}|y_l|) \cap B(y_k, \tfrac{1}{2}|y_k|) \neq \emptyset\} \leq C_n N(n)$$

for some constant C_n. Working in local charts, and using partitions of unity, the above construction provides the required covering near the boundary of a manifold.

We refer the reader to [12, Section 5] for a justification of the remaining hypotheses of Theorem 5.2.

Theorem 6.4. *Let $k \in \{2, \dots, m\}$, $\alpha \in (0,1)$, $g_0 \in \mathcal{M}_{k+2,\alpha}$, $K_0 \in \mathsf{C}^{-1}_{k+2,\alpha}$, with*

$$x|K_0|_{g_0} + x^2|\nabla K_0|_{g_0} \to_{x\to 0} 0.$$

For $\sigma \geq 0$ when $n > 3$ or $\sigma > 0$ any non-empty connected component of

$$\mathcal{S}_{J_0,\rho_0} = \Big\{(Q,h) \in \Lambda^{\sigma-1}_{k+2,\alpha} \times \mathcal{A}^{\sigma}_{k+2,\alpha}, \\ (J,\rho)(K_0+Q, g_0+h) = (J_0,\rho_0), \operatorname{Ker} P^*_{K_0+Q,g_0+h} = \{0\}\Big\}$$

is an embedded submanifold of $\Lambda^{\sigma-1}_{k+2,\alpha} \times \mathcal{A}^{\sigma}_{k+2,\alpha}$.

Here the kernel of the operator $P^*_{K_0+Q,g_0+h}$ is, by elliptic regularity, a subspace of $(\mathring{\mathsf{H}}_1^{-\sigma-n+2} \times \mathring{\mathsf{H}}_2^{-\sigma-n+2}) \cap (C^{k+2}(\bar{M}) \times C^{k+2}(\bar{M}))$.

For further reference we note the following result.

Proposition 6.5. *Suppose that $(K_0, g_0) \in (C^{k+2,\alpha} \times C^{k+2,\alpha})(M)$, $k \geq 2$, $\alpha \in (0,1)$, and let $\Omega \subset M$ be a domain with smooth boundary and compact closure. For all $s \neq (n+1)/2, (n+3)/2$, the image of the linearization P, at (K_0, g_0), of the constraints map, when defined on $(\Lambda^{-s+1}_{k+2,\alpha} \times \Lambda^{-s+2}_{k+2,\alpha})(\Omega)$,* is

$$(x^{n-2s}\mathcal{K}_0^{\perp_{g_0}}) \cap (\Lambda^{-s}_{k+1,\alpha} \times \Lambda^{-s}_{k,\alpha}).$$

Here $\mathcal{K}_0$ is the space of KIDs which are in $\mathring{\mathsf{H}}_1^{s-n} \times \mathring{\mathsf{H}}_2^{s-n} \subset (L^2 \times L^2)(\Omega, x^{-2s+n}\,\mathrm{d}\mu_{g_0})$, and orthogonality is taken in $(L^2 \times L^2)(\Omega, x^{-2s+n}\,\mathrm{d}\mu_{g_0})$. In other words, the image of P is

$$\Big\{(J,\rho) \in \Lambda^{-s}_{k+1,\alpha} \times \Lambda^{-s}_{k,\alpha} \text{ such that } \langle (J,\rho), (Y,N)\rangle_{(L^2\oplus L^2)(\Omega,\,\mathrm{d}\mu_{g_0})} = 0 \\ \text{for all } (Y,N) \in H_1^{s-n} \times H_2^{s-n} \text{ satisfying } P^*(Y,N) = 0\Big\}.$$

Further $P^{-1}(0) \subset \Lambda^{-s+1}_{k+2,\alpha} \times \Lambda^{-s+2}_{k+2,\alpha}$ splits.

Proof. The proof of this result is essentially contained in that of Theorem 6.4; the restriction on the constant σ there arises from the requirement that the full non-linear constraint map be well defined, but this restriction is not needed for the linearized problem. We note that a closed space complementing $P^{-1}(0)$ is provided by $\mathrm{Im}\,(\Phi^2\psi^2 P^*_\epsilon)$, for ϵ small enough (compare the arguments in Appendix A), and that P restricted to this space is an isomorphism onto $\mathrm{Im}(P)$. □

Alternative useful weights are the exponential ones, for those we will use Proposition B.5 to obtain the needed regularization. We take the following weight functions[11]

$$\phi = x^2, \qquad \psi = x^n\,\mathrm{e}^{-s/x}, \qquad \varphi = x^{2n}\,\mathrm{e}^{-s/x}.$$

With those choices we note that

$$\mathcal{A}^{k+2,\alpha}_{x^2,x^n\,\mathrm{e}^{-s/x},x^{2n}\,\mathrm{e}^{-s/x}} \subset \psi^2\phi^2(H_{\phi,\psi}\cap C_{\phi,\varphi}) = x^4(\mathring{H}^2_{x^2,x^{-n}\,\mathrm{e}^{s/x}}\cap C^{k+2,\alpha}_{x^2,\mathrm{e}^{s/x}}).$$

We define the space

$$\mathcal{M}^{\exp}_{k,\alpha} = \Big\{ g \text{ metric uniformly equivalent to } \gamma,\\ g-\gamma \in C^{k,\alpha}_{x^2,1}, |\nabla^{(l)}_\gamma(g-\gamma)|_\gamma = \mathrm{O}(x^{-2l}),\ \ 0\le l\le 2\Big\}.$$

Using the results in Section 5 of [12] one now has the following theorem.

Theorem 6.6. *Let $k\in\{2,\dots,m\}$, $\alpha\in(0,1)$, $g_0\in\mathcal{M}^{\exp}_{k+2,\alpha}$, $K_0\in x^{-2}C^{k+2,\alpha}_{x^2,1}$, with*

$$x^2|K_0|_{g_0} + x^4|\nabla K_0|_{g_0} \to_{x\to 0} 0.$$

For $s>0$ any non-empty connected component of

$$\mathcal{S}_{J_0,\rho_0} = \Big\{(Q,h)\in x^2\Lambda^{2,k+2,\alpha}_{x^2,x^{-n}\,\mathrm{e}^{s/x},\mathrm{e}^{s/x}}\times\mathcal{A}^{k+2,\alpha}_{x^2,x^n\,\mathrm{e}^{-s/x},x^{2n}\,\mathrm{e}^{-s/x}},\\ (J,\rho)(K_0+Q,g_0+h)=(J_0,\rho_0),\ \mathrm{Ker}\,P^*_{K_0+Q,g_0+h}=\{0\}\Big\}$$

is an embedded submanifold of $x^2\Lambda^{2,k+2,\alpha}_{x^2,x^{-n}\,\mathrm{e}^{s/x},\mathrm{e}^{s/x}}\times\mathcal{A}^{k+2,\alpha}_{x^2,x^n\,\mathrm{e}^{-s/x},x^{2n}\,\mathrm{e}^{-s/x}}$.

Here the kernel of the operator $P^*_{K_0+Q,g_0+h}$ is a subspace of $\mathring{H}^1_{x^2,x^n\,\mathrm{e}^{-s/x}}\times\mathring{H}^2_{x^2,x^n\,\mathrm{e}^{-s/x}}$; but elliptic regularity shows that elements of the kernel are classically differentiable in the interior, and it is standard to show that they are in $C^{k+2}(\bar M)\times C^{k+2}(\bar M)$.

6.4. Conformally compactifiable manifolds

In this section we apply Theorem 5.2 to the case where M is a conformally compactifiable manifold (with a compact conformal boundary at infinity), as in Section 6 of [12]. Let γ be

[11] Theorem 5.9 and Proposition 5.10 of [12] provide δg in $\psi^2\phi^2(\mathring{H}_{\phi,\psi}\cap C_{\phi,\varphi}) = x^4(\mathring{H}_{x^2,x^{-n}\,\mathrm{e}^{s/x}}\cap C_{x^2,\mathrm{e}^{s/x}})$, which leads to the choices of the weights above.

$C^{m+4,\alpha}$ conformally compactifiable, so that $\gamma = x^{-2}\bar\gamma$, with x—defining function for the conformal boundary ∂M, and $\bar\gamma$—a Riemannian metric on $\bar M$ of $C^{m+4,\alpha}(\bar M)$ differentiability class. In that context, it is natural to define

$$\mathring{\mathcal{H}}^s_k := \mathring{H}^k_{1,x^{-s}}(\gamma) = \mathring{\mathsf{H}}^s_k(\bar\gamma),$$

$$\mathcal{C}^s_{k,\alpha} := C^{k,\alpha}_{x,x^{-s}}(\bar\gamma) = \mathsf{C}^s_{k,\alpha}(\bar\gamma),$$

$$\Lambda^s_{k,\alpha} = \Lambda^{2,k+2,\alpha}_{1,x^{-s},x^{-s}}(\gamma) = \mathcal{C}^s_{k,\alpha} \cap \mathring{\mathcal{H}}^s_2,$$

and for $t \geq 0$,

$$\mathcal{A}^t_{k+2,\alpha} = \mathcal{A}^{k+2,\alpha}_{1,x^t,x^t}(\gamma) \subset \mathcal{C}^t_{k+2,\alpha} \subset \mathcal{C}^0_{k+2,\alpha}.$$

The space $\mathcal{M}_{k,\alpha}$ is defined similarly to (6.3), but both the background γ and the function spaces involved are different now:

$$\mathcal{M}_{k,\alpha} = \Big\{ g \text{ metric uniformly equivalent to } \gamma, \\ g-\gamma \in \mathcal{C}^0_{k,\alpha}, |\nabla^{(l)}_\gamma (g-\gamma)|_\gamma = o(1), \quad 0 \leq l \leq 2 \Big\}.$$

The same locally uniform covering as in the preceding section can be used here, so the smoothing operator of Appendix B.1 applies, leading to

Theorem 6.7. *Let $k \in \{2, \dots, m\}$ and $\alpha \in (0,1)$. Let $g_0 \in \mathcal{M}_{k,\alpha}$, $K_0 = \lambda_0 g_0 + L_0$ with $L_0, \lambda_0 \in \mathcal{C}^0_{k+2,\alpha}$, $|L_0|_\gamma \to_{x\to 0} 0$, $|\nabla L_0|_\gamma \to_{x\to 0} 0$. Let $t \geq 0$, $t \notin \{(n-3)/2, (n-1)/2, (n+1)/2\}$. Then any connected component of the set*

$$\mathcal{S}_{J_0,\rho_0} = \Big\{ (Q,h) \in \Lambda^t_{k+2,\alpha} \times \mathcal{A}^t_{k+2,\alpha}, \\ (J,\rho)(K_0+Q, g_0+h) = (J_0,\rho_0), \operatorname{Ker} P^*_{K_0+Q,g_0+h} = \{0\} \Big\}$$

is a submanifold of $\Lambda^t_{k+2,\alpha} \times \mathcal{A}^t_{k+2,\alpha}$. For $0 \leq t < (n+1)/2$ the kernel condition is automatically satisfied.

Here the kernel of $P^*_{K_0+Q,g_0+h}$ is that for a map from $\mathring{\mathcal{H}}^{-t}_1 \times \mathring{\mathcal{H}}^{-t}_2$ to $\mathring{\mathcal{H}}^{-t}_0 \times \mathring{\mathcal{H}}^{-t}_0$; this can also be reformulated in terms of classical differentiability in appropriately weighted spaces.

Remarks similar to those following Corollary 6.3 concerning the value of t apply here.

Remark 6.8. For $(Q,h) \in \mathcal{S}_{J_0,\rho_0}$ we have that $|Q|_\gamma = o(1)$ and $|\nabla Q|_\gamma = o(1)$.

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Appendix A. Submanifolds, foliations

The following is a variation of an argument in [17].

Lemma A.1. *Let E, F and G be three Banach spaces, and let u (resp. v) be a map defined from a neighborhood of 0 in E (resp. F) to F (resp. G) such that $u(0) = 0$ (resp. $v(0) = 0$) and which is differentiable at 0. We also assume that $Dv(0) \circ Du(0)$ is an isomorphism from E to G. Then the set $v^{-1}(0)$ is a submanifold of F in a neighborhood of 0.*

Proof. For $x \in F$,

$$x = \underbrace{Du(0) \circ [Dv(0) \circ Du(0)]^{-1} \circ Dv(0)(x)}_{\in \operatorname{Im} Du(0)} + \underbrace{x - Du(0) \circ [Dv(0) \circ Du(0)]^{-1} \circ Dv(0)(x)}_{\in \operatorname{Ker} Dv(0)}.$$

(It easily follows that $F = \operatorname{Im} Du(0) \oplus \operatorname{Ker} Dv(0)$, with both summands closed.) As $Dv(0) \circ Du(0)$ is an isomorphism, the inverse function theorem shows that $v \circ u$ is a diffeomorphism in a neighborhood of 0, so for $x \in F$ close to zero we have

$$x = \underbrace{u \circ [v \circ u]^{-1} \circ v(x)}_{\in \operatorname{Im} u} + x - u \circ [v \circ u]^{-1} \circ v(x).$$

Let us define a map from a neighborhood of zero in F to F by

$$f(x) = x + u \circ [v \circ u]^{-1} \circ v(x) - Du(0) \circ [Dv(0) \circ Du(0)]^{-1} \circ Dv(0)(x).$$

One clearly has

$$Df(0) = \mathrm{Id},$$

and the inverse function theorem shows that f is a diffeomorphism in a neighborhood of zero. We also have

$$x \in v^{-1}(0) \Leftrightarrow f(x) \in \operatorname{Ker} Dv(0)$$

(for the "$\Leftarrow$" part we use the fact that $Dv(0) \circ u \circ [v \circ u]^{-1}$ is an isomorphism near zero), so f provides the required map modeling $v^{-1}(0)$ on a linear space. □

Lemma A.1 shows how to straighten-up a level set of v; one can similarly show existence of foliations by level sets.

Lemma A.2. *Let E, F and G three Banach spaces, L a linear continuous map from E to F, v a map defined from a neighborhood of a point x_0 in F to G, continuously differentiable*

near x_0. We assume that $Dv(x_0) \circ L$ is an isomorphism from F to G. Then there exists a neighborhood V of $y_0 = v(x_0)$ in G such that the collection of level sets of v,

$$\{x \in F, v(x) = y\}_{y\in V},$$

is a foliation of a neighborhood U of x_0 in F.

Proof. We have that $Dv(x) \circ L$ is a diffeomorphism for $x \in F$ close to x_0, one then has as in Lemma A.1 that

$$F = \operatorname{Im} L \oplus \operatorname{Ker} Dv(x),$$

and one easily checks that $\operatorname{Im} L$ is closed (recall that all maps are continuous). So $Dv(x)$ is surjective and its kernel splits. From [25, p. 21] the map v is then a submersion near x_0. In particular, again from [25, p. 20], there exist U neighborhood of x_0 in F, V neighborhood of y_0 in G and two isomorphisms $\varphi : U \to U_1 \times U_2$ (U_1 and U_2 open subset of some Banach spaces) and $\psi : V \to V_2$ (V_2 open subset of some Banach spaces with $U_2 \subset V_2$) such that

$$\psi \circ v \circ \varphi^{-1} : U_1 \times U_2 \to V_2$$

is the projection on the second axis. This gives the desired foliation of U. □

Appendix B. Two weighted smoothing operators

In this appendix we will show how to define smoothing operators as needed in the body of the paper; this will require a set of conditions on the functions ϕ and φ, compatible with the usual settings of interest in general relativity. The technique of Appendix B.2 seems to be somewhat simpler than that of Appendix B.1, and does not require any covering conditions. However, covering conditions arise naturally when regularizing functions in weighted Sobolev classes, therefore it seemed of interest to us to present both methods.

B.1. Smoothing with locally uniformly finite coverings

Throughout this appendix we assume that the manifold M is an open subset of $\mathbb{R}^n$, equipped with an Euclidean metric (which is of course not the physical space metric we are interested in), and we will be regularizing functions. The regularization can then be applied to tensor fields on more general manifolds by using coordinate patches, partitions of unity, and usual covering arguments.

We assume that ϕ and φ verify (4.1)–(4.3). For all $p \in M$, we denote by B_p, the open ball of center p with radius $\phi(p)/2$. We require that[12] for all $p \in M$,

$$B(p, \phi(p)) \subset M. \tag{B.1}$$

[12] Eq. (B.1) can be replaced by the weaker condition that there exists $\mu > 0$ such that for all $p \in M$ we have $B(p, \mu\phi(p)) \subset M$, as changing ϕ to $\mu\phi$ for a positive constant μ leads to equivalent norms. So, e.g. in the asymptotically flat case, one actually has to replace the weight $\phi = r$ for $r \geq R$ by $\phi = r/2R$. Any such rescaling leads to obvious changes in the hypotheses needed for the covering arguments below.

For $\rho \in [1, \infty)$ we shall use the following notation:

$$B_i^\rho = B\left(p_i, \frac{\phi(p_i)}{\rho}\right).$$

Our next restriction is that the manifold can be covered by a countable collection of balls B_i^8,

$$M = \bigcup_{i=1}^{\infty} B_i^8, \tag{B.2}$$

such that there exists an $N \in \mathbb{N}$ so that for all $i \in \mathbb{N}$,

$$\#\{j, B_i^2 \cap B_j^2 \neq \emptyset\} \leq N. \tag{B.3}$$

For $p \in M$, we set

$$\varphi_p : B(0, \tfrac{1}{2}) \ni z \mapsto p + \phi(p)z \in B_p.$$

This implies that for all functions u on M and all multi-indices γ we have

$$\partial_z^\gamma (u \circ \varphi_p) = \phi(p)^{|\gamma|} (\partial^\gamma u) \circ \varphi_p.$$

Using (4.2) and (4.3) it is easy to see that we have the following lemma.

Lemma B.1. *For $\rho \in [2, 8]$ the following norms on $C^{k,\alpha}_{\phi,\varphi}(M)$ are equivalent*:

$$\|u\|_{C^{k,\alpha}_{\phi,\varphi}(M)} \sim \sup_{i\in\mathbb{N}} \|u\|_{C^{k,\alpha}_{\phi,\varphi}(B_i^\rho)} \sim \sup_{i\in\mathbb{N}} \|u\|_{C^{k,\alpha}_{\phi(p_i),\varphi(p_i)}(B_i^\rho)} \sim \sup_{i\in\mathbb{N}} \|u \circ \varphi_{p_i}\|_{C^{k,\alpha}_{1,\varphi(p_i)}(B(0,1/\rho))}.$$

We now construct a convenient partition of unity.

Lemma B.2. *There exists a partition of unity*

$$\sum_{i=1}^{\infty} \zeta_i = 1$$

with smooth functions $\zeta_i \geq 0$, and $\zeta_i > 0$ outside B_i^4, such that for all $l \in \mathbb{N}$ and $\alpha \in (0, 1)$ there exists a constant $C(l, \alpha)$ so that for all $i \in \mathbb{N}$,

$$\|\zeta_i\|_{C^{l,\alpha}_{\phi,1}(M)} \leq C(l, \alpha).$$

Proof. Let χ be a smooth non-negative function on $\mathbb{R}^n$ such that $\chi = 1$ on $B(0, 1/8)$ and $\chi = 0$ outside $B(0, 1/4)$. We define

$$\chi_i := \chi \circ \varphi_{p_i}^{-1}, \qquad \zeta_i := \frac{\chi_i}{\sum_{j=1}^{\infty} \chi_j}.$$

Let us show that the sum in the definition above is well defined and greater than 1. If $x \in B_i^4$, there exists at most N balls B_j^4, with N given by (B.3), such that $x \in B_j^4$. Since χ_j

has support in B_j^4, the sum at x is over a finite set. If $x \in B_i^8$ then $\chi_i(x) = 1$ thus the sum is not less than 1. If $x \in B_i^4 \setminus B_i^8$, from (B.2), x must be in some B_j^8 thus $\chi_j(x) = 1$ and then, again, the sum is greater than or equal to 1.

Now, χ_i has compact support so is in $C^{l,\alpha}_{\phi,1}(M)$, and from Lemma B.1

$$\|\chi_i\|_{C^{l,\alpha}_{\phi,1}(M)} = \|\chi_i\|_{C^{l,\alpha}_{\phi,1}(B_i^4)} \leq C'\|\chi_i \circ \varphi_{p_i}\|_{C^{l,\alpha}_{1,1}(B(0,1/4))} = C'\|\chi\|_{C^{l,\alpha}_{1,1}(B(0,1/4))} =: C_1,$$

where C_1 depends upon l and α but does not depend upon i. So we have that

$$\left\|\sum_{j=1}^{\infty}\chi_j\right\|_{C^{l,\alpha}_{\phi,1}(B_i^2)} \leq \sum_{\{j, B_i^2\cap B_j^2\neq\emptyset\}} \|\chi_j\|_{C^{l,\alpha}_{\phi,1}(M)} \leq NC_1,$$

and thus

$$\left\|\sum_{j=1}^{\infty}\chi_j\right\|_{C^{l,\alpha}_{\phi,1}(M)} \leq NC_1.$$

Finally, as the sum is greater than or equal to 1 it is easy to see that the ζ_i's satisfy the desired properties. □

Let θ by any smooth strictly positive function on $\mathbb{R}^n$, with support in $B(0, 1)$ and such that

$$\int_{\mathbb{R}^n}\theta = 1.$$

For $\varepsilon > 0$ we set

$$\theta_\varepsilon(x) = \frac{1}{\varepsilon^n}\theta\left(\frac{x}{\varepsilon}\right).$$

For u a function on M and $\varepsilon > 0$, we define

$$u_i = \zeta_i u, \qquad \hat{u}_i = u_i \circ \varphi_{p_i},$$

$$\hat{u}_{i,\varepsilon} = \theta_\varepsilon * \hat{u}_i, \qquad u_{i,\varepsilon} = \hat{u}_{i,\varepsilon} \circ \varphi_{p_i}^{-1}, \qquad u_\varepsilon = \sum_{i=1}^{\infty} u_{i,\varepsilon}$$

Proposition B.3. *Let $u \in C^{k,\alpha}_{\phi,\varphi}(M)$. For all $\varepsilon \in (0, 1/4)$ and all $m \in \mathbb{N}$ we have $u_\varepsilon \in C^m_{\phi,\varphi}(M)$. Further, u_ε converges to u in $C^{k,\alpha}_{\phi,\varphi}(M)$ as ε goes to zero.*

Proof. First remark that as the u_i's have support in B_i^4, then for $\varepsilon < 1/4$ the functions $u_{i,\varepsilon}$ have support in B_i^2. It follows that on B_i^2 we have

$$u_\varepsilon = \sum_{\{j, B_i^2\cap B_j^2\neq\emptyset\}} u_{j,\varepsilon}.$$

Lemmas B.1 and B.2 together with standard properties of convolution in $\mathbb{R}^n$ imply

$$\begin{aligned}\|u_\varepsilon\|_{C^{k,\alpha}_{\phi,\varphi}(B_i^2)} &\le \sum_{\{j,B_i^2\cap B_j^2\neq\emptyset\}} \|u_{j,\varepsilon}\|_{C^{k,\alpha}_{\phi,\varphi}(B_i^2)} = \sum_{\{j,B_i^2\cap B_j^2\neq\emptyset\}} \|u_{j,\varepsilon}\|_{C^{k,\alpha}_{\phi,\varphi}(B_i^2\cap B_j^2)}\\ &\le \sum_{\{j,B_i^2\cap B_j^2\neq\emptyset\}} \|u_{j,\varepsilon}\|_{C^{k,\alpha}_{\phi,\varphi}(B_j^2)} \le C \sum_{\{j,B_i^2\cap B_j^2\neq\emptyset\}} \|\hat u_{j,\varepsilon}\|_{C^{k,\alpha}_{\phi(p_j),\varphi(p_j)}(B(0,1/2))}\\ &\le C \sum_{\{j,B_i^2\cap B_j^2\neq\emptyset\}} \|\hat u_j\|_{C^{k,\alpha}_{\phi(p_j),\varphi(p_j)}(B(0,1/2))}\\ &\le C' \sum_{\{j,B_i^2\cap B_j^2\neq\emptyset\}} \|u_j\|_{C^{k,\alpha}_{\phi,\varphi}(B_i^2)} \le NC''C(k,\alpha)\|u\|_{C^{k,\alpha}_{\phi,\varphi}(M)}.\end{aligned}$$

In particular $u_\varepsilon \in C^{k,\alpha}_{\phi,\varphi}(M)$, with norm uniformly bounded in ε. Let us now show that in fact u_ε is also in $C^{k+l}_{\phi,\varphi}(M)$ for any $l \ge 0$. First, we have

$$\begin{aligned}\sup_{B_i^2} |\varphi\phi^{k+l}\partial^{k+l}u_{i,\varepsilon}| &\le C\sup_{B_i^2} |\varphi(p_i)\phi^{k+l}(p_i)\partial^{k+l}u_{i,\varepsilon}| = C\sup_{B(0,1/2)} |\varphi(p_i)\partial^{k+l}\hat u_{i,\varepsilon}|\\ &\le C\|\partial^l\theta_\varepsilon\|_{L^1(B(0,1/2))}\|\varphi(p_i)\partial^k\hat u_i\|_{L^\infty(B(0,1/2))}\\ &= C\frac{1}{\varepsilon^l}\|\partial^l\theta\|_{L^1(\mathbb{R}^n)}\|\varphi(p_i)\partial^k\hat u_i\|_{L^\infty(B(0,1/2))}\\ &\le C'\frac{C_l}{\varepsilon^l}\|\varphi\phi^k\partial^k u_i\|_{L^\infty(B_i^2)} \le C''\frac{C_l}{\varepsilon^l}\|u\|_{C^k_{\phi,\varphi}(M)}.\end{aligned}$$

Thus we have

$$\sup_{B_i^2} |\varphi\phi^{k+l}\partial^{k+l}u_\varepsilon| \le \sum_{\{j,B_i^2\cap B_j^2\neq\emptyset\}} \sup_{B_i^2\cap B_j^2} |\varphi\phi^{k+l}\partial^{k+l}u_{j,\varepsilon}| \le NC'''\frac{C_l}{\varepsilon^l}\|u\|_{C^k_{\phi,\varphi}(M)},$$

and then, using Lemma B.1,

$$\sup_M |\varphi\phi^{k+l}\partial^{k+l}u_\varepsilon| \le NC'''\frac{C_l}{\varepsilon^l}\|u\|_{C^k_{\phi,\varphi}(M)},$$

which gives $u_\varepsilon \in C^{k+l}_{\phi,\varphi}(M)$ for any $l \ge 0$.

Let us pass now to the proof that u_ε converges to u in $C^{k,\alpha}_{\phi,\varphi}(M)$. We have

$$\begin{aligned}\|u_{i,\varepsilon}-u_i\|_{C^{k,\alpha}_{\phi,\varphi}(B_i^2)} &\le C\|\hat u_{i,\varepsilon}-\hat u_i\|_{C^{k,\alpha}_{1,\varphi(p_i)}(B(0,1/2))}\\ &\le C'\varepsilon^\alpha\|\hat u_i\|_{C^{k,\alpha}_{1,\varphi(p_i)}(B(0,1/2))} \le C''\varepsilon^\alpha\|u\|_{C^{k,\alpha}_{\phi,\varphi}(M)}.\end{aligned}$$

Thus as $u_\varepsilon - u = \sum_{i=1}^\infty (u_{i,\varepsilon}-u_i)$, on B_i^2, we have

$$\|u_\varepsilon-u\|_{C^{k,\alpha}_{\phi,\varphi}(B_i^2)} \le \sum_{\{j,B_i^2\cap B_j^2\neq 0\}} \|u_{j,\varepsilon}-u_j\|_{C^{k,\alpha}_{\phi,\varphi}(B_i^2\cap B_j^2)} \le NC''\varepsilon^\alpha\|u\|_{C^{k,\alpha}_{\phi,\varphi}(B_i^2)},$$

then finally

$$\|u_\varepsilon - u\|_{C^{k,\alpha}_{\phi,\varphi}(M)} \leq NC'\varepsilon^\alpha \|u\|_{C^{k,\alpha}_{\phi,\varphi}(M)}.$$

□

B.2. Smoothing using a weighted convolution

For the exponential weights considered in Section 6.3 the uniform covering condition of the previous section seems to be awkward to verify directly, if true. It is therefore convenient to proceed differently. In what follows we will assume that M is an open domain in $\mathbb{R}^n$, this can again be gotten rid of by using partitions of unity and passing to local charts. As in the preceding section, we assume that ϕ and φ satisfy (4.1)–(4.3) and that (B.1) holds.

Lemma B.4. *The following norms on $C^{k,\alpha}_{\phi,\varphi}(M)$ are equivalent*:

$$\|u\|_{C^{k,\alpha}_{\phi,\varphi}(M)} \sim \sup_{p\in M} \|u\|_{C^{k,\alpha}_{\phi,\varphi}(B_p)} \sim \sup_{p\in M} \|u\|_{C^{k,\alpha}_{\phi(p),\varphi(p)}(B_p)}.$$

For $u \in C^{k,\alpha}_{\phi,\varphi}(M)$ and for $\epsilon \in (0, 1/2)$, we define the smooth function on M (here θ is as in the preceding section):

$$\begin{aligned}\tilde{u}_\varepsilon(x) &= \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n \phi(x)^n} \theta\left(\frac{x-y}{\varepsilon\phi(x)}\right) u(y)\, \mathrm{d}^n y \\ &= \int_{y\in B(x,\varepsilon\phi(x))} \frac{1}{\varepsilon^n \phi(x)^n} \theta\left(\frac{x-y}{\varepsilon\phi(x)}\right) u(y)\, \mathrm{d}^n y = \int_{z\in B(0,1)} \theta(z) u(x-\varepsilon\phi(x)z)\, \mathrm{d}^n z.\end{aligned}$$

Proposition B.5. *Let $u \in C^{k,\alpha}_{\phi,\varphi}(M)$. For all $\varepsilon \in (0, 1/2)$ and all $m \in \mathbb{N}$ we have $\tilde{u}_\varepsilon \in C^m_{\phi,\varphi}(M)$. Further, $\tilde{u}_\varepsilon$ converges to u in $C^{k,\alpha}_{\phi,\varphi}(M)$ as ε goes to zero.*

Proof. We first show that $\tilde{u}_\varepsilon \in C^{k,\alpha}_{\phi,\varphi}(M)$ with norm bounded independently of ε. We have

$$|\varphi(x)\tilde{u}_\varepsilon(x)| \leq \sup_{y\in B^n(x,\varepsilon\phi(x))} |\varphi(x)u(y)| \leq C\|u\|_{C^{k,\alpha}_{\phi,\varphi}(M)},$$

where the last inequality comes from Lemma B.4. For the first derivatives, we have

$$\partial_i \tilde{u}_\varepsilon(x) = \int_{z\in B^n(0,1)} \theta(z)\partial_j u(x-\varepsilon\phi(x)z)(\delta^j_i - \varepsilon\partial_i\phi(x)z^j)\, \mathrm{d}^n z.$$

So from Eq. (2.5) and Lemma B.4 we have

$$|\varphi(x)\phi(x)\partial_i\tilde{u}_\varepsilon(x)| \leq C \sup_{y\in B^n(x,\varepsilon\phi(x))} |\varphi(x)\phi(x)\partial u(y)| \leq C'\|u\|_{C^{k,\alpha}_{\phi,\varphi}(M)}.$$

It should be clear that similar inequalities are true for the kth derivatives and for the Hölder quotients, leading to

$$\|\tilde{u}_\varepsilon\|_{C^{k,\alpha}_{\phi,\varphi}(M)} \leq C\|u\|_{C^{k,\alpha}_{\phi,\varphi}(M)}.$$

We leave it as an exercise to the reader to show in a manner similar to that in the preceding section that $\tilde{u}_\varepsilon$ is in fact in $C^{k+l}_{\phi,\varphi}(M)$ for all $l \in \mathbb{N}$.

Writing

$$\tilde{u}_\varepsilon(x) - u(x) = \int_{z\in B^n(0,1)} \theta(z)[u(x - \varepsilon\phi(x)z) - u(x)]\,\mathrm{d}^n z,$$

one similarly shows that

$$\|\tilde{u}_\varepsilon - u\|_{C^{k,\alpha}_{\phi,\varphi}(M)} \leq C\|u\|_{C^{k,\alpha}_{\phi,\varphi}(M)}\varepsilon^\alpha,$$

so that $\tilde{u}_\varepsilon$ converges to u in $C^{k,\alpha}_{\phi,\varphi}(M)$ when ε goes to zero, as required. □

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